# Review Problems
# For Theoretical Mechanics

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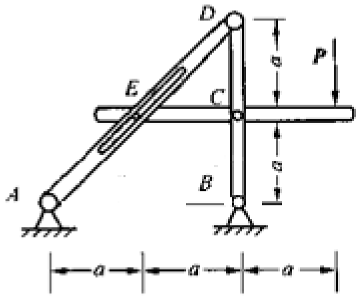
**Edited by The Department of Mechanics**

2019.11

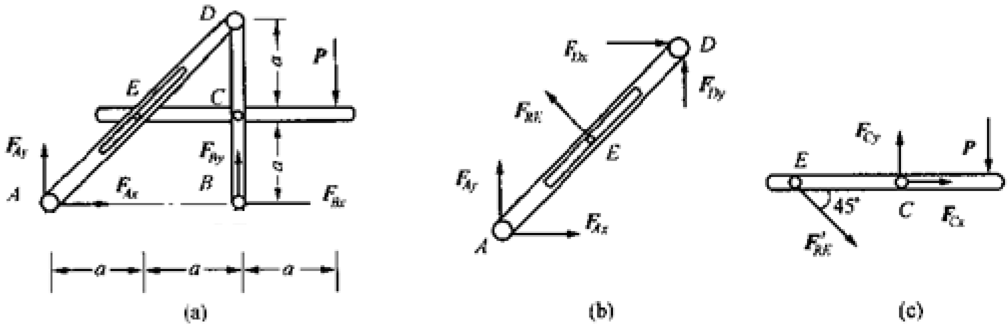
# Statics

**Ep 1.** Known: P, a, the weight and friction of each member are not counted; Find the constraint reaction at A and B.




**SOLUTION:**

Firstly, take the whole as the research object, and its force diagram is shown in the figure (a).




Ep 1

[Analysis]: There are altogether four unknown forces in this force diagram, all of which are required constraint forces. There are only three independent equilibrium equations, so it is impossible to solve all of them. But by taking the moment of A or B, we can solve for $\mathbf{F_{AY}}$ or $\mathbf{F_{BY}}$ first.

From $\quad \sum \mathrm{m}_B(\vec{F}) = 0 \qquad -2a \times F_{AY} - P \times a = 0 \qquad$ we get: $\quad F_{AY} = -\dfrac{1}{2}P$

$\sum Y = 0 \qquad F_{AY} + F_{BY} - P = 0 \qquad$ we get: $\quad F_{BY} = \dfrac{3}{2}P$

Take the rod EC, and its force is shown in Figure 3.1 (c), from

$$\sum m_C(\vec{F}) = 0 \qquad F'_{RE} \sin 45^\circ \times a - P \times a = 0$$

We obtain $\qquad F'_{RE} = \sqrt{2}P$

Take bar AED, and its force diagram is shown in Fig. 3.1 (b), from

$$\sum m_D(\vec{F}) = 0 \qquad 2a \times F_{Ax} - 2a \times F_{Ay} - \sqrt{2}a \times F_{RE} = 0$$

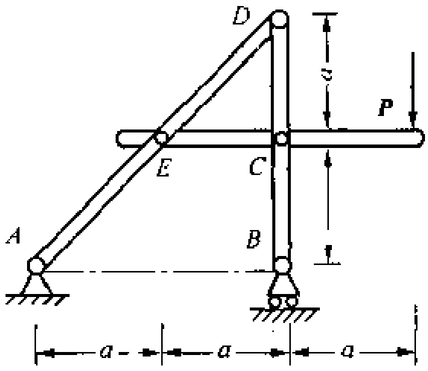
We get $\qquad F_{Ax} = \dfrac{P}{2}$

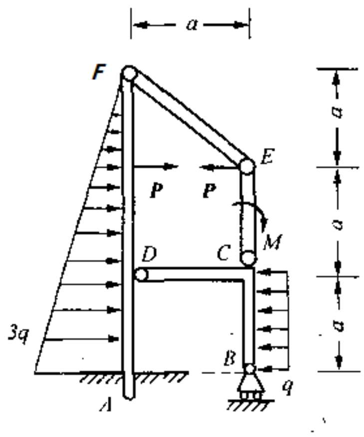
For the overall force figure 3.1 (a), from

$\sum X = 0 \qquad F_{Ax} + F_{Bx} = 0$

we get $\qquad F_{Bx} = -\frac{P}{2}$

**Exercise:** Known: P, a, the weight and friction of each member are not counted;
Calculate the constraint reaction at A, B and E.

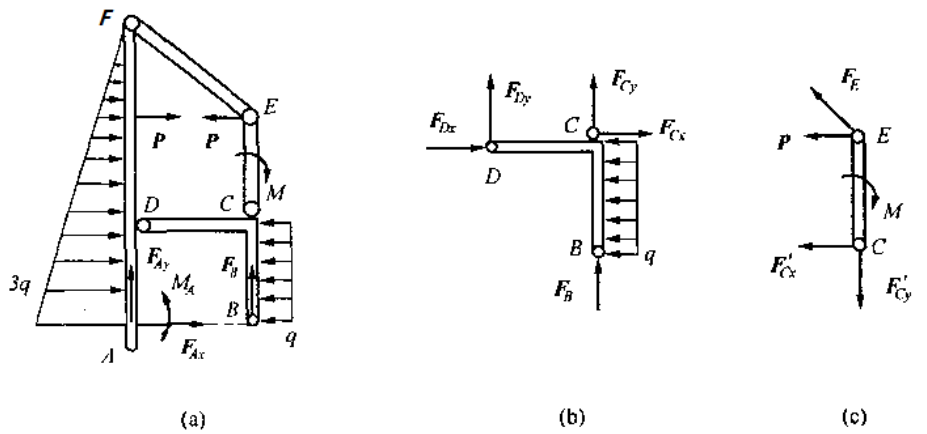


**Ep 2**. Known：P=10kN，q=2kN/m, M=2kN •m, ɑ=1m, the weight and friction of each member are not counted;
Calculate the constraint reaction at A.

**SOLUTION:**

Firstly, take the whole as the research object, and its stress is shown in Figure 2 (a).



Ep 2

$$\sum X = 0 \qquad F_{Ax} + \frac{1}{2} \times 3q \times 3a - qa = 0$$

We get $$F_{Ax} = -7kN$$

Take the bar CE, and its force diagram is shown in FIG. 3.2 (c), from

$$\sum m_c(\vec{F}) = 0 \qquad F_E \cos 45^\circ \times a + P \times a - M = 0$$

We get $$F_E \cos 45^\circ = -8kN$$

From $$\sum Y = 0 \qquad F_E \sin 45^\circ - F'_{Cy} = 0$$

get $$F'_{Cy} = -8kN$$

Take the component BCD, and its force diagram is shown in Figure 3.2 (b), from

$$\sum m_D(\vec{F}) = 0 \qquad F_B \times a + F_{Cy} \times a - qa \times \frac{a}{2} = 0$$

Get $$F_B = 9kN$$

For the whole, from

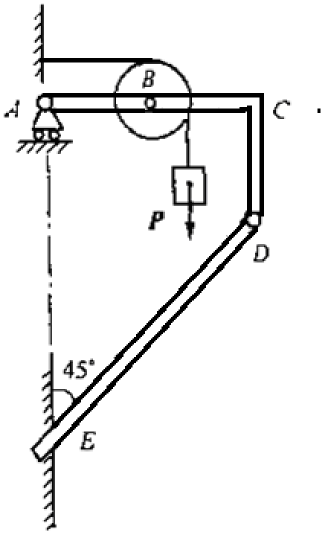
$$\sum Y = 0 \qquad F_{Ay} + F_B = 0$$

Get $$F_{Ay} = -9kN$$

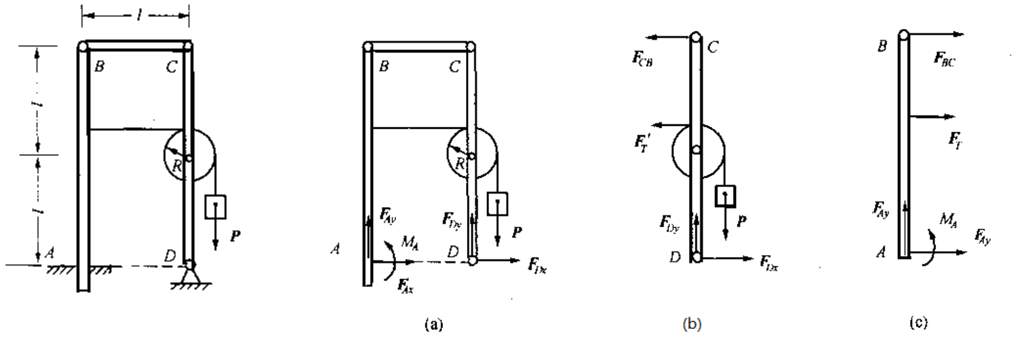
From $\sum m_A(\vec{F})=0$ $M_A + aF_B - M + qa\times\frac{a}{2} - \frac{1}{2}\times 3q\times 3a\times a = 0$

get $M_A = 1kN\cdot m$

Exercise：Known：AB=BC=CD=ɑ. The material weight is P, and the weight of pulley and each bar is not counted. Find the constraint reaction at E.



**Ep 3.** Known: P, l, R, Each bar and pulley shall be excluded; Find the constraint reaction at fixed end A.




Ep 3

**SOLUTION:**

Take the CD rod (including the pulley) as the research object, and its force is shown in Figure (b), from

$$\sum m_D(\vec{F})=0 \qquad F_{CB}\times 2l+F_T'(l+R)-PR=0$$

We get $$F_{CB}=-\frac{P}{2}$$

Take rod AB and its force diagram is shown in Figure (c), from

$$\sum X=0 \qquad F_{Ax}+F_{BC}+F_T=0$$

We get $$F_{Ax}=-\frac{P}{2}$$

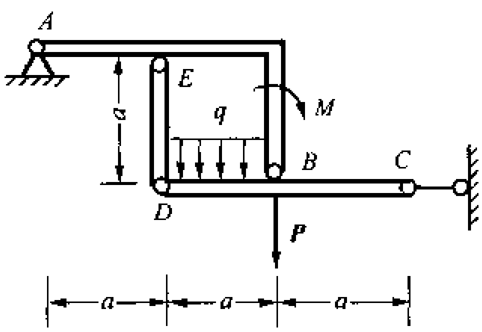
From $\sum Y=0$ $\qquad F_{Ay}=0$

$$\sum m_A(\vec{F})=0 \qquad M_A-F_T(l+R)-F_{BC}\times 2l=0$$

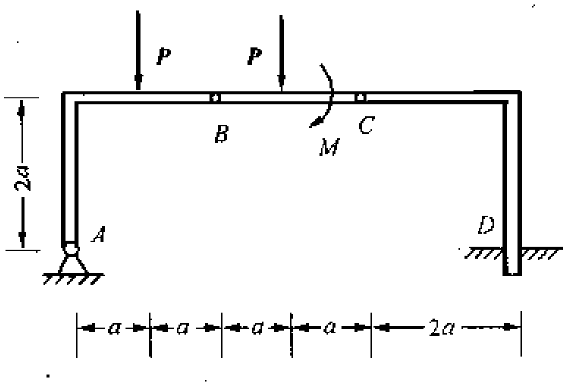
We get $$M_A=PR$$

**Exercise:** Known: $q=500N/m, P=2000N, M=500Nm, a=1m$, Not counting the weight of each pole.

Calculate: the force of AB bar on CD bar at B.



**Ep 4.** Known: $P$, $a$, $M = Pa$, Not counting the weight of each pole.
Find the constraint reaction at support A and D.



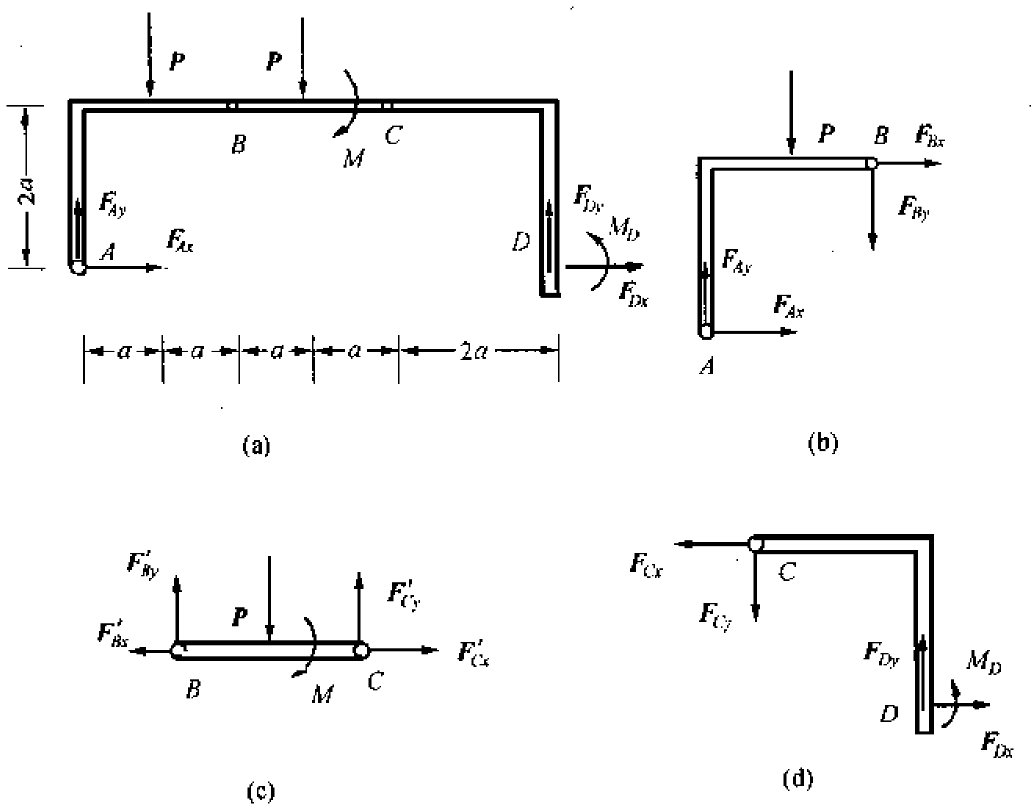

**SOLUTION:**

Firstly, BC bar is taken as the object of study, and its force diagram is shown in Figure (c), from

$$\sum m_C(\vec{F}) = 0$$

$$-F'_{By} \times 2a + Pa - M = 0$$

We get $F'_{By} = 0$




Ep 4

Finally, take the CD rod as the research object, and the force is shown in Figure (d),

$$\sum X = 0 \qquad -F_{Cx} + F_{Dx} = 0 \qquad \text{get} \qquad F_{Dx} = -\frac{P}{2}$$

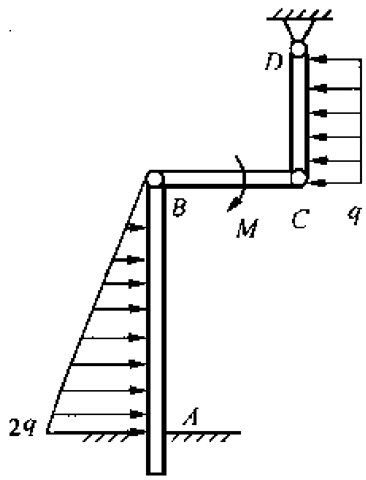
$$\sum Y = 0 \qquad F_{Dy} - F_{Cy} = 0 \qquad \text{get} \qquad F_{Dy} = P$$

$$\sum m_D(\vec{F})=0 \qquad M_D+2aF_{Cx}+2aF_{cy}=0 \qquad \text{get} \quad M_D=-Pa$$

**Exercise:**

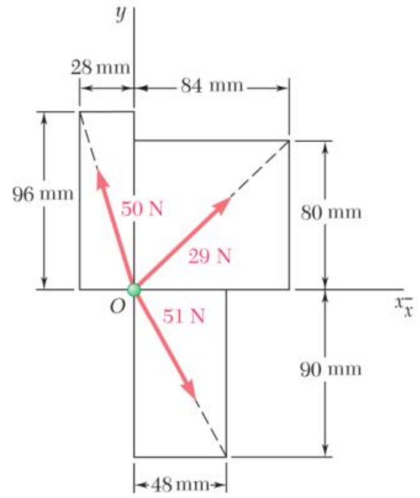
Known: AB=2BC=2CD=2m， $q$=2000N/m，M=500Nm, Each rod is homogeneous and its weight per unit length is 500N/m.

Calculate: Constraint reaction at A.

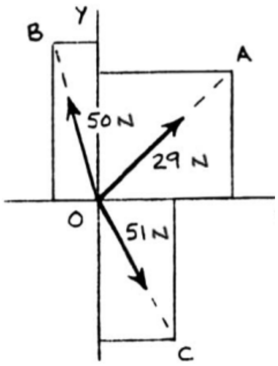


**Ep 5.**

Determine the $x$ and $y$ components of each of the forces shown.




**SOLUTION**




Compute the following distances:

$$OA = \sqrt{(84)^2 + (80)^2}$$
$$= 116 \text{ mm}$$
$$OB = \sqrt{(28)^2 + (96)^2}$$
$$= 100 \text{ mm}$$
$$OC = \sqrt{(48)^2 + (90)^2}$$
$$= 102 \text{ mm}$$

29-N Force:

$$F_x = +(29 \text{ N})\frac{84}{116} \qquad F_x = +21.0 \text{ N}$$

$$F_y = +(29 \text{ N})\frac{80}{116} \qquad F_y = +20.0 \text{ N}$$

50-N Force:

$$F_x = -(50 \text{ N})\frac{28}{100} \qquad F_x = -14.00 \text{ N}$$

$$F_y = +(50 \text{ N})\frac{96}{100} \qquad F_y = +48.0 \text{ N}$$

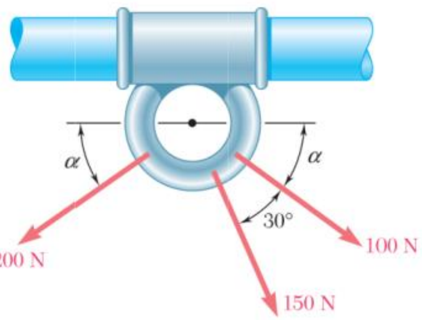
51-N Force:

$$F_x = +(51 \text{ N})\frac{48}{102} \qquad F_x = +24.0 \text{ N}$$

$$F_y = -(51 \text{ N})\frac{90}{102} \qquad F_y = -45.0 \text{ N}$$

**Ep 6.**

Knowing that $\alpha = 35°$, determine the resultant of the three forces shown.



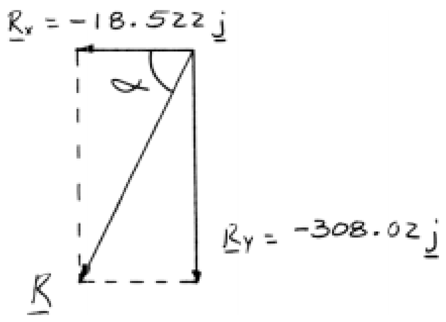

**SOLUTION**

100-N Force:
$$F_x = +(100\text{ N})\cos 35° = +81.915\text{ N}$$
$$F_y = -(100\text{ N})\sin 35° = -57.358\text{ N}$$

150-N Force:
$$F_x = +(150\text{ N})\cos 65° = +63.393\text{ N}$$
$$F_y = -(150\text{ N})\sin 65° = -135.946\text{ N}$$

200-N Force:
$$F_x = -(200\text{ N})\cos 35° = -163.830\text{ N}$$
$$F_y = -(200\text{ N})\sin 35° = -114.715\text{ N}$$

| Force | $x$ Comp. (N) | $y$ Comp. (N) |
|---|---|---|
| 100 N | +81.915 | −57.358 |
| 150 N | +63.393 | −135.946 |
| 200 N | −163.830 | −114.715 |
| | $R_x = -18.522$ | $R_y = -308.02$ |




$$\mathbf{R} = R_x\mathbf{i} + R_y\mathbf{j}$$
$$= (-18.522\text{ N})\mathbf{i} + (-308.02\text{ N})\mathbf{j}$$

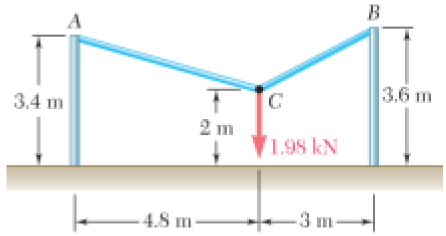
$$\tan\alpha = \frac{R_y}{R_x}$$
$$= \frac{308.02}{18.522}$$
$$\alpha = 86.559°$$

$$R = \frac{308.02\text{ N}}{\sin 86.559}$$

$\mathbf{R} = 309\text{ N} \measuredangle 86.6°$

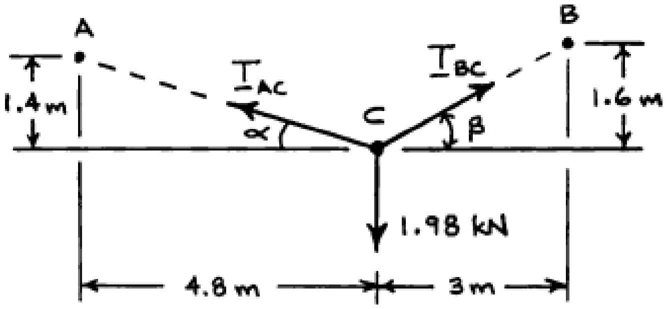
**Ep 7.**

Two cables are tied together at $C$ and loaded as shown. Determine the tension ($a$) in cable $AC$, ($b$) in cable $BC$.



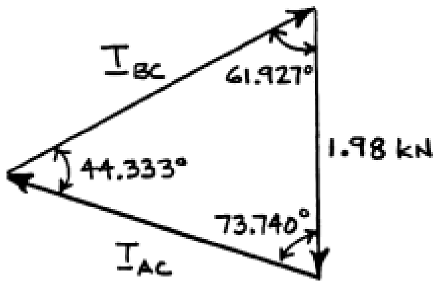

**SOLUTION**

**Free-Body Diagram**




$$\tan\alpha = \frac{1.4}{4.8}$$
$$\alpha = 16.2602^\circ$$
$$\tan\beta = \frac{1.6}{3}$$
$$\beta = 28.073^\circ$$

**Force Triangle**




Law of sines:

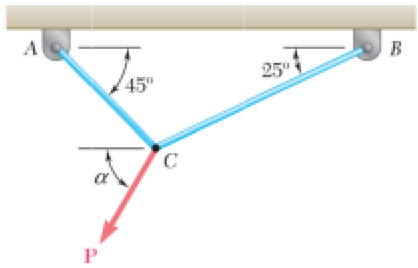
$$\frac{T_{AC}}{\sin 61.927^\circ} = \frac{T_{BC}}{\sin 73.740^\circ} = \frac{1.98 \text{ kN}}{\sin 44.333^\circ}$$

($a$) $$T_{AC} = \frac{1.98 \text{ kN}}{\sin 44.333^\circ}\sin 61.927^\circ \qquad T_{AC} = 2.50 \text{ kN}$$

($b$) $$T_{BC} = \frac{1.98 \text{ kN}}{\sin 44.333^\circ}\sin 73.740^\circ \qquad T_{BC} = 2.72 \text{ kN}$$

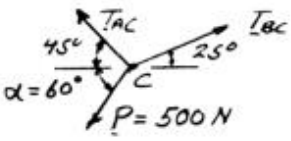
**Ep 8.**

Two cables are tied together at $C$ and are loaded as shown. Knowing that $\mathbf{P}$ = 500 N and $\alpha$ = 60°, determine the tension in ($a$) in cable $AC$, ($b$) in cable $BC$.

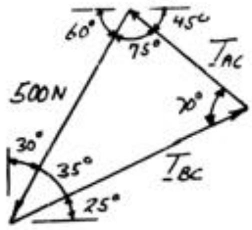



**SOLUTION**

**Free-Body Diagram**




**Force Triangle**



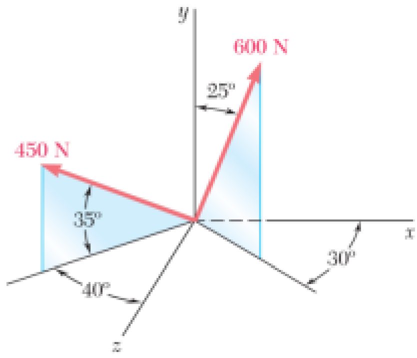

Law of sines:
$$\frac{T_{AC}}{\sin 35°} = \frac{T_{BC}}{\sin 75°} = \frac{500 \text{ N}}{\sin 70°}$$

($a$)
$$T_{AC} = \frac{500 \text{ N}}{\sin 70°} \sin 35° \qquad T_{AC} = 305 \text{ N}$$

($b$)
$$T_{BC} = \frac{500 \text{ N}}{\sin 70°} \sin 75° \qquad T_{BC} = 514 \text{ N}$$

**Ep 9.**

Determine (*a*) the *x*, *y*, and *z* components of the 600-N force, (*b*) the angles $\theta_x$, $\theta_y$, and $\theta_z$ that the force forms with the coordinate axes.




**SOLUTION**

(*a*)

$$F_x = (600\text{ N})\sin 25^\circ \cos 30^\circ$$

$$F_x = 219.60\text{ N} \qquad F_x = 220\text{ N}$$

$$F_y = (600\text{ N})\cos 25^\circ$$

$$F_y = 543.78\text{ N} \qquad F_y = 544\text{ N}$$

$$F_z = (380.36\text{ N})\sin 25^\circ \sin 30^\circ$$

$$F_z = 126.785\text{ N} \qquad F_z = 126.8\text{ N}$$

(*b*)

$$\cos\theta_x = \frac{F_x}{F} = \frac{219.60\text{ N}}{600\text{ N}} \qquad \theta_x = 68.5^\circ$$

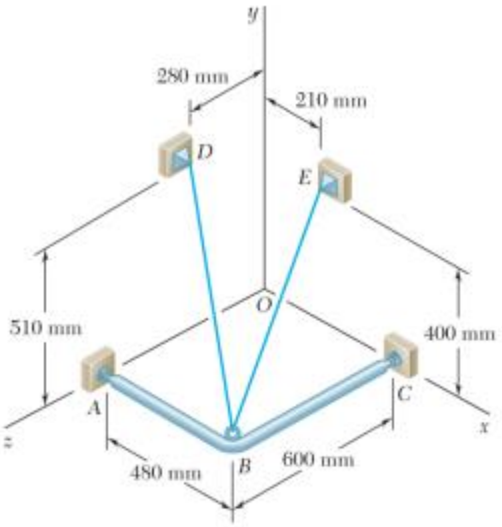
$$\cos\theta_y = \frac{F_y}{F} = \frac{543.78\text{ N}}{600\text{ N}} \qquad \theta_y = 25.0^\circ$$

$$\cos\theta_z = \frac{F_z}{F} = \frac{126.785\text{ N}}{600\text{ N}} \qquad \theta_z = 77.8^\circ$$

**Ep 10.**

For the frame and cable of Problem 2.85, determine the components of the force exerted by the cable on the support at $E$.

**PROBLEM 2.85** A frame $ABC$ is supported in part by cable $DBE$ that passes through a frictionless ring at $B$. Knowing that the tension in the cable is 385 N, determine the components of the force exerted by the cable on the support at $D$.




**SOLUTION**

$$\overrightarrow{EB} = (270 \text{ mm})\mathbf{i} - (400 \text{ mm})\mathbf{j} + (600 \text{ mm})\mathbf{k}$$

$$EB = \sqrt{(270 \text{ mm})^2 + (400 \text{ mm})^2 + (600 \text{ mm})^2}$$

$$= 770 \text{ mm}$$

$$\mathbf{F} = F\boldsymbol{\lambda}_{EB}$$

$$= F\frac{\overrightarrow{EB}}{EB}$$

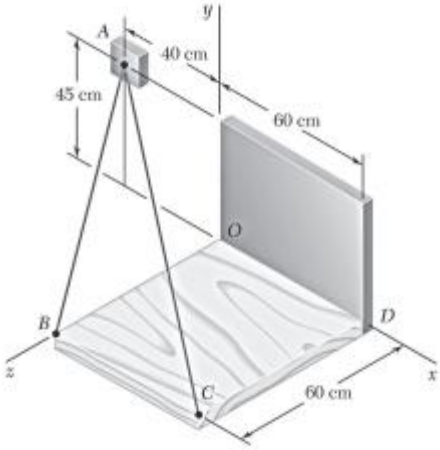
$$= \frac{385 \text{ N}}{770 \text{ mm}}[(270 \text{ mm})\mathbf{i} - (400 \text{ mm})\mathbf{j} + (600 \text{ mm})\mathbf{k}]$$

$$\mathbf{F} = (135 \text{ N})\mathbf{i} - (200 \text{ N})\mathbf{j} + (300 \text{ N})\mathbf{k}$$

$$F_x = +135.0 \text{ N}, \quad F_y = -200 \text{ N}, \quad F_z = +300 \text{ N}$$

**Ep 11.**

Knowing that the tension is 425 N in cable $AB$ and 510 N in cable $AC$, determine the magnitude and direction of the resultant of the forces exerted at $A$ by the two cables.




**SOLUTION**

$$\overrightarrow{AB} = (40\text{ cm})\mathbf{i} - (45\text{ cm})\mathbf{j} + (60\text{ cm})\mathbf{k}$$

$$AB = \sqrt{(40\text{ cm})^2 + (45\text{ cm})^2 + (60\text{ cm})^2} = 85\text{ cm}$$

$$\overrightarrow{AC} = (100\text{ cm})\mathbf{i} - (45\text{ cm})\mathbf{j} + (60\text{ cm})\mathbf{k}$$

$$AC = \sqrt{(100\text{ cm})^2 + (45\text{ cm})^2 + (60\text{ cm})^2} = 125\text{ cm}$$

$$\mathbf{T}_{AB} = T_{AB}\boldsymbol{\lambda}_{AB} = T_{AB}\frac{\overrightarrow{AB}}{AB} = (425\text{ N})\left[\frac{(40\text{ cm})\mathbf{i} - (45\text{ cm})\mathbf{j} + (60\text{ cm})\mathbf{k}}{85\text{ cm}}\right]$$

$$\mathbf{T}_{AB} = (200\text{ N})\mathbf{i} - (225\text{ N})\mathbf{j} + (300\text{ N})\mathbf{k}$$

$$\mathbf{T}_{AC} = T_{AC}\boldsymbol{\lambda}_{AC} = T_{AC}\frac{\overrightarrow{AC}}{AC} = (510\text{ N})\left[\frac{(100\text{ cm})\mathbf{i} - (45\text{ cm})\mathbf{j} + (60\text{ cm})\mathbf{k}}{125\text{ cm}}\right]$$

$$\mathbf{T}_{AC} = (408\text{ N})\mathbf{i} - (183.6\text{ N})\mathbf{j} + (244.8\text{ N})\mathbf{k}$$

$$\mathbf{R} = \mathbf{T}_{AB} + \mathbf{T}_{AC} = (608\text{ N})\mathbf{i} - (408.6\text{ N})\mathbf{j} + (544.8\text{ N})\mathbf{k}$$

Then: $R = 912.92\text{ N}$ $\qquad R = 913\text{ N}$

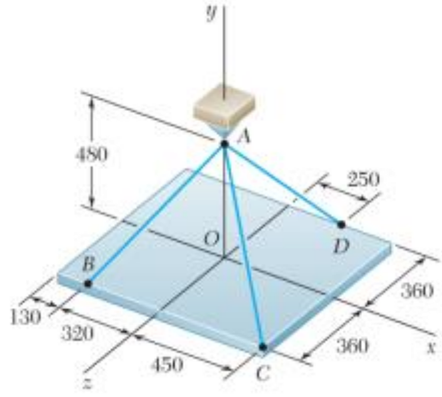
and $\cos\theta_x = \dfrac{608\text{ N}}{912.92\text{ N}} = 0.66599$ $\qquad \theta_x = 48.2°$

$\cos\theta_y = \dfrac{-408.6\text{ N}}{912.92\text{ N}} = -0.44757$ $\qquad \theta_y = 116.6°$

$\cos\theta_z = \dfrac{544.8\text{ N}}{912.92\text{ N}} = 0.59677$ $\qquad \theta_z = 53.4°$

## Ep 12.

For the rectangular plate of Problems 2.109 and 2.110, determine the tension in each of the three cables knowing that the weight of the plate is 792 N.




**SOLUTION**

See Problem 2.109 for the figure and the analysis leading to the linear algebraic Equations (1), (2), and (3) below. Setting $P = 792$ N gives:

$$-\frac{8}{17}T_{AB} + 0.6T_{AC} + \frac{5}{13}T_{AD} = 0 \qquad (1)$$

$$-\frac{12}{17}T_{AB} - 0.64T_{AC} - \frac{9.6}{13}T_{AD} + 792\text{ N} = 0 \qquad (2)$$

$$\frac{9}{17}T_{AB} + 0.48T_{AC} - \frac{7.2}{13}T_{AD} = 0 \qquad (3)$$

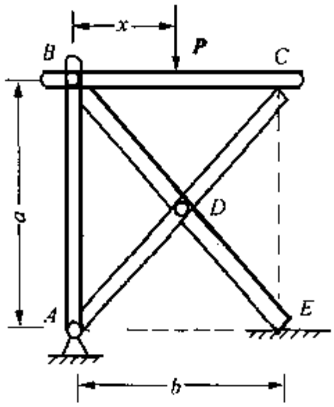
Solving Equations (1), (2), and (3) by conventional algorithms gives

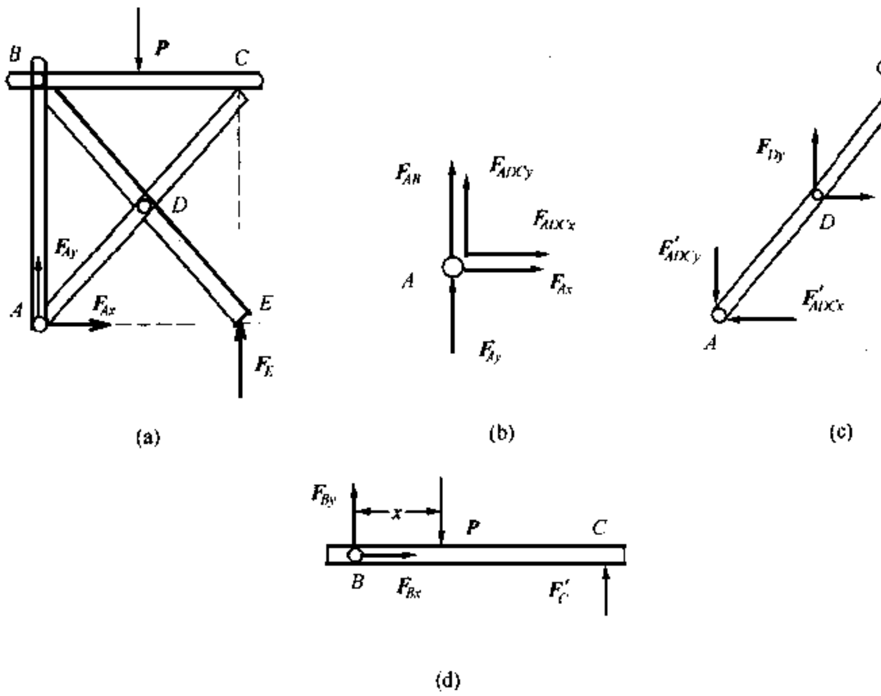
$T_{AB} = 510.00$ N $\qquad T_{AB} = 510$ N

$T_{AC} = 56.250$ N $\qquad T_{AC} = 56.2$ N

$T_{AD} = 536.25$ N $\qquad T_{AD} = 536$ N

**Ep 13.** Known: Dimensions A, B, P acting on the BC bar can be translated with X, C, E is smooth contact, the dead weight of each bar is not calculated, pin A, B penetrates each member.
Find out: AB bar stress, and explain whether AB bar stress and X.





Ep 13

**SOLUTION:**

Take the whole as the research object, and the force is shown in Figure (a),

$$\sum X = 0 \qquad F_{Ax} = 0$$

$$\sum m_E(\vec{F}) = 0 \qquad P(b-x) - F_{Ay}b = 0 \qquad \text{get} \quad F_{Ay} = \frac{b-x}{b}P$$

Pin A was taken as the object of study, and its stress was shown in Figure (b),

$$\sum X = 0 \qquad F_{Ax} + F_{ADCx} = 0 \qquad \text{get} \quad F_{ADCx} = 0$$

BC bar was taken as the research object, and its stress was shown in Figure (d),

$$\sum m_B(\vec{F})=0 \qquad F_C'b-Px=0 \qquad 得\ F_C'=\frac{x}{b}P$$

ADC bar is taken as the research object, and its force is shown in Figure (c),

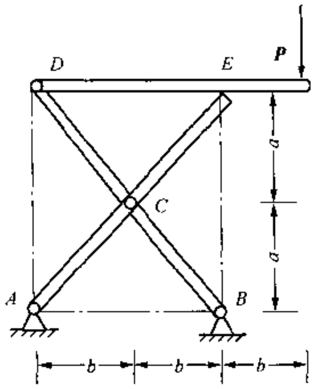
$$\sum m_D(\vec{F})=0 \qquad F'_{ADCy}\times\frac{b}{2}-F'_{ADCx}\times\frac{a}{2}-F_C\times\frac{b}{2}=0 \quad \text{get} \quad F'_{AXCy}=\frac{x}{b}P$$

Then pin A is taken as the research object, and there are:

$$\sum Y=0 \qquad F_{AB}+F_{Ay}+F_{ADCy}=0$$

That is $F_{AB}+\frac{b-x}{b}P+\frac{x}{b}P=0$ get $F_{AB}=-P$

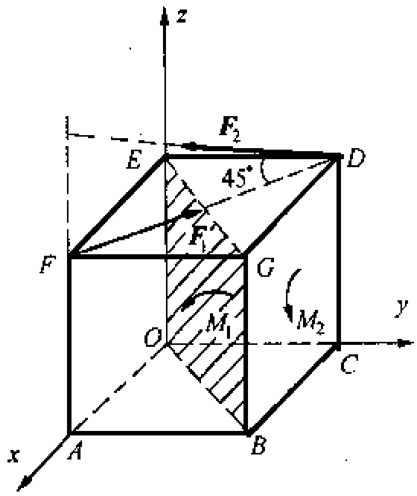
That is, the AB bar is under the action of pressure P, independent of position X.




**Exercise:** Known: P、a、b, the point E is smooth contact, and the weight of each rod is not counted. Calculate: the force of BCD bar on ACE bar at C.

**Ep 14.** The cuboid's side length is A, and its acting force are $F_1$ and $F_2$. The acting position is shown in Figure 6. The couple acting surface with a moment of $M_1$ is in OBGE plane, while the couple acting surface with a moment of $M_2$ is in BCDG plane.

Calculate: the projection of each force on the *x*,*y*, *z* axes and the simplified result of the moment of *x*, *y*, *z* axes and the force system toward O point.




Ep 14

**SOLUTION:**

Projection:

$$F_{1x} = -\frac{\sqrt{2}}{2}F_1 \qquad F_{1y} = \frac{\sqrt{2}}{2}F_1 \qquad F_{1z} = 0$$

$$F_{2x} = F_2 \cos 45^\circ \cos 45^\circ = \frac{1}{2}F_2$$

$$F_{2y} = -F_2 \cos 45^\circ \sin 45^\circ = -\frac{1}{2}F_2$$

$$F_{2z} = F_2 \sin 45^\circ = \frac{\sqrt{2}}{2}F_2$$

The moments of the *x*, *y*, *z* axes of each force and its couple are respectively

$$m_x(\vec{F}_1) = -\frac{\sqrt{2}}{2}F_1 a \qquad m_y(\vec{F}_1) = -\frac{\sqrt{2}}{2}F_1 a \qquad m_z(\vec{F}_1) = \frac{\sqrt{2}}{2}F_1 a$$

$$m_x(\vec{F}_2) = \frac{1}{2}F_2 a + \frac{\sqrt{2}}{2}F_2 a \qquad m_y(\vec{F}_2) = \frac{1}{2}F_2 a \qquad m_z(\vec{F}_2) = -\frac{1}{2}F_2 a$$

$$M_{1x} = M_1 \cos 45^\circ = \frac{\sqrt{2}}{2}M_1 \qquad M_{1y} = -M_1 \sin 45^\circ = -\frac{\sqrt{2}}{2}M_1 \qquad M_{1z} = 0$$

$$M_{2x} = 0 \qquad M_{2y} = M_2 \qquad M_{2z} = 0$$

The simplified principal vector principal moment of the force system towards O point is

$$R'_x = \sum X = -\frac{\sqrt{2}}{2}F_1 + \frac{1}{2}F_2$$

$$R'_y = \sum Y = \frac{\sqrt{2}}{2}F_1 - \frac{1}{2}F_2$$

$$R'_z = \sum Z = \frac{\sqrt{2}}{2} F_2$$

$$\vec{R}' = R'_x \vec{i} + R'_y \vec{j} + R'_z \vec{k}$$

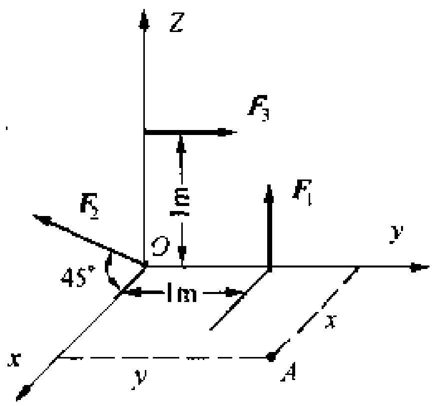
$$M_x = \sum m_x(\vec{F}) = -\frac{\sqrt{2}}{2} F_1 a + \frac{1}{2} F_2 a + \frac{\sqrt{2}}{2} F_2 a + \frac{\sqrt{2}}{2} M_1$$

$$M_y = \sum m_y(\vec{F}) = -\frac{\sqrt{2}}{2} F_1 a + \frac{1}{2} F_2 a - \frac{\sqrt{2}}{2} M_1 + M_2$$

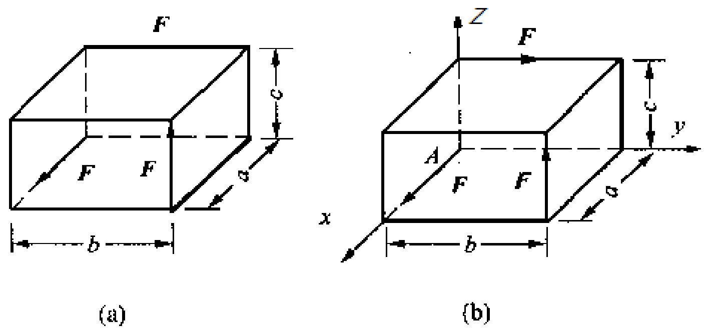
$$M_z = \sum m_z(\vec{F}) = \frac{\sqrt{2}}{2} F_1 a - \frac{1}{2} F_2 a$$

$$\vec{M}_o = M_x \vec{i} + M_y \vec{j} + M_z \vec{k}$$

**Exercise:** The position of three forces $\vec{F}_1, \vec{F}_2, \vec{F}_3$ is shown in the right figure, $\vec{F}_1 // x$, $\vec{F}_3 // y$, $\vec{F}_2$ is in the *Oxz* plane, $F_1 = 100N$, $F_2 = 100\sqrt{2}N, F_3 = 200N$, if the principal moment of a simplified vector in the *xy* plane is in the same square, calculate the coordinates of point A and the principal vector and principal moment simplified to point A.



**Ep 15.** As is shown in Figure 7(a), three disjoint and uneven edges along the cuboid act on three equal forces F. Find out: what relationship should edge A, B and C have so that the force system can be reduced to a force.



Ep 15

**SOLUTION:**

Select A as the simplification center, and establish the coordinate system as shown in Figure (b). The simplification result is as follows:

$$\vec{R}' = F\vec{i} + F\vec{j} + F\vec{k}$$

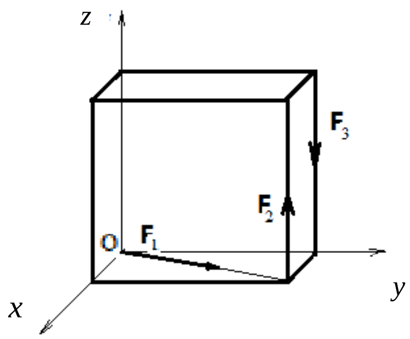
$$\vec{M}_A = (Fb - Fc)\vec{i} - Fa\vec{j}$$

When $\vec{R}' \perp \vec{M}_A$, the final simplified result is a net force, therefore:

$$\overline{R}' \bullet \vec{M}_A = 0 \qquad \text{that is} \qquad F^2(b-c) - F^2 a = 0$$

Thus: $a = b - c$

**Exercise:** As shown in the figure on the right, the side length of the cube is a, and the forces acting on it are all three forces of size F, which are simplified to point O. Judge whether the final simplification result is a resultant force.



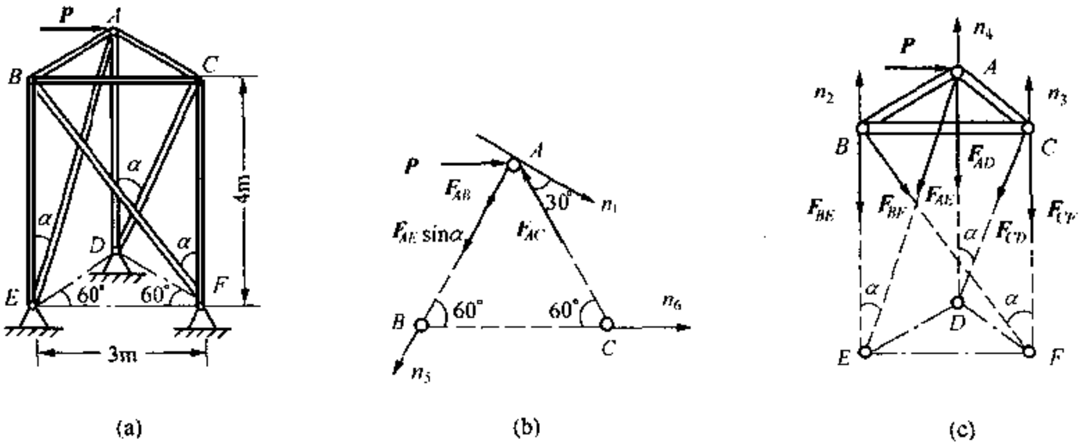
**Ep 16.** As shown in Figure 8 (a), the frame is composed of nine rods regardless of self-weight, AB=BC=CA=3m, BE=CF=AD=4m, ABC forms a horizontal plane, BE, CF and AD rods are perpendicular, and force P is parallel to rod BC.
Calculate: The internal force of each bar.

**SOLUTION:**

Take node A as the research object, and the force is shown in Figure (b). $\vec{n}_1 \perp \overline{AB}$, the axial force of AB, AD and AE bar is projected on it to be zero.

$$\sum F_{n1} = 0 \qquad P\cos 30^\circ - F_{AC}\cos 30^\circ = 0$$

We get $\qquad F_{AC} = P$




Ep 16

Take ABC as the research object, and the force is shown in Figure. (c), from

$$\sum m_{n2}(\vec{F}) = 0 \qquad F_{CD}\sin\alpha \times 3\cos 30^\circ - P \times 3\cos 30^\circ = 0$$

get $\qquad F_{CD} = \frac{5}{3}P$

$$\sum m_{n3}(\vec{F}) = 0 \qquad F_{AE}\sin\alpha \times 3\cos 30^\circ - P \times 3\cos 30^\circ = 0$$

get $\qquad F_{AE} = \frac{5}{3}P$

$$\sum m_{n4}(\vec{F}) = 0 \qquad F_{BF}\sin\alpha \times 3\cos 30^\circ = 0$$

get $\qquad F_{BF} = 0$

For the intersection point A, see Figure (b, c).

$$\sum F_{n4} = 0 \qquad -F_{AD} - F_{AE}\cos\alpha = 0$$

get $$F_{AD} = -\frac{4}{3}P$$

$$\sum F_{n5} = 0 \qquad F_{AB} + F_{AE}\sin\alpha - P\cos 60^\circ - F_{AC}\cos 60^\circ = 0$$

get $$F_{AB} = 0$$

For the intersection point B, see Figure (b, c).

From $\sum F_{n2} = 0 \qquad -F_{BE} - F_{BF}\cos\alpha = 0$

Get $$F_{BE} = 0$$

$$\sum F_{n6} = 0 \qquad F_{BC} + F_{BF}\sin\alpha + F_{AB}\cos 60^\circ = 0$$

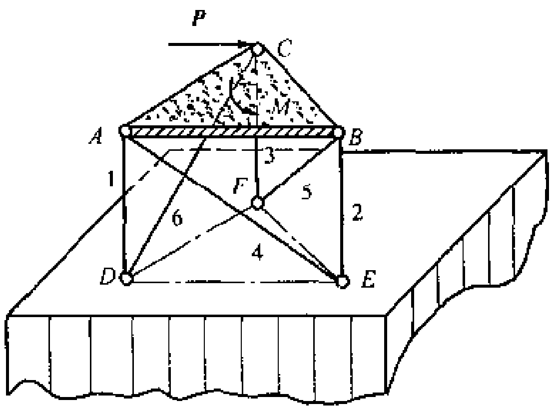
Get $$F_{BC} = 0$$

For the intersection point C, see Figure (b, c)

$$\sum F_{n3} = 0 \qquad -F_{CF} - F_{CD}\cos\alpha = 0$$

Get $$F_{CF} = -\frac{4}{3}P$$

**Exercise:** The equilateral triangle ABC, with weight P and side length A, is supported in the horizontal position by three non-weight lead-bar 1, 2, 3 and three non-weight inclined straight bars 4, 5 and 6 at horizontal angle with ball hinge. A force couple is acted on the plate surface, its moment is $M = \frac{\sqrt{3}}{2}Pa$. At point C, a force P parallel to side AB is applied. To calculate the internal force of each bar.



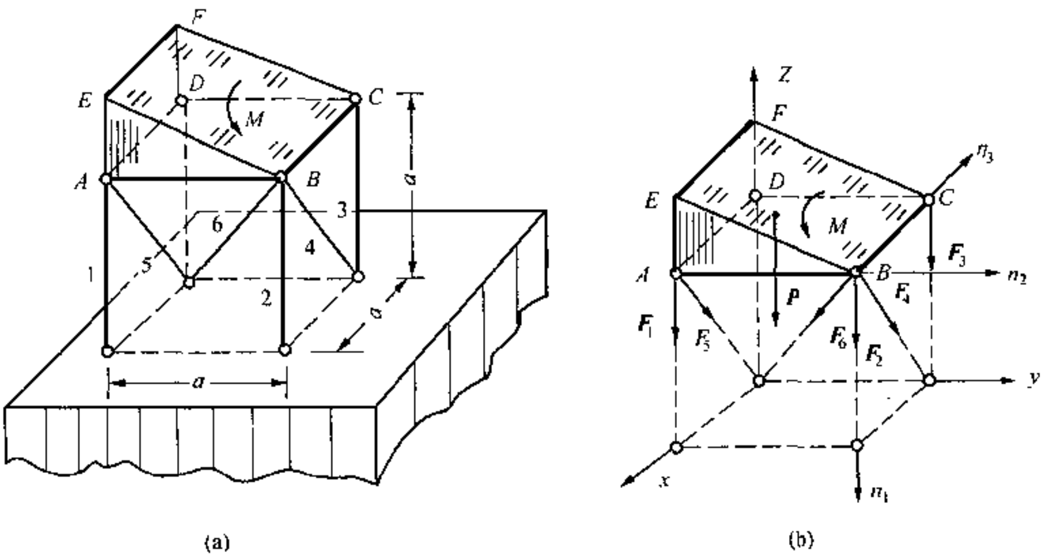
**Ep 17.** The weight P of homogeneous rectangular prism ABCDEF is1 500N, $\angle ABE = 30^\circ$, a couple M is applied in the BCEF plane, M=500Nm, connected by six weightless rods connected by ball hinge, as shown in Figure 9, a=1m.Calculate the internal force of each rod.

**SOLUTION:**

The force is shown in Figure (b)

$$\sum Y = 0 \qquad -F_6 \frac{\sqrt{2}}{\sqrt{3}} \cos 45^\circ = 0$$

Get $\qquad F_6 = 0$




Ep 17

From $\qquad \sum m_z(\vec{F}) = 0 \qquad F_4 \cos 45^\circ \times a + M \sin 60^\circ = 0$

Get $\qquad F_4 = -250\sqrt{6}N$

From $\qquad \sum m_{n1}(\vec{F}) = 0 \qquad F_5 \cos 45^\circ \times a - M \sin 60^\circ = 0$

Get $\qquad F_4 = 250\sqrt{6}N$

From $\qquad \sum m_{n2}(\vec{F}) = 0 \qquad -F_3 \times a - P \times \frac{a}{2} + M \cos 60^\circ = 0$

Get $\qquad F_3 = -500N$

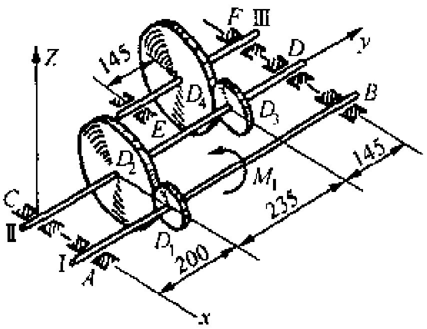
From $\qquad \sum m_{n3}(\vec{F}) = 0 \qquad -(F_1 a + F_5 \cos 45^\circ \times a + P \times \frac{2a}{3}) = 0$

Get $\qquad F_5 = -(500 + 250\sqrt{3})N$

From $\sum m_x(\vec{F}) = 0$ $\quad - F_2 \times a - F_4 \cos 45^\circ \times a - F_3 \times a - P \times \frac{a}{3} = 0$

Get $F_5 = -(500 - 250\sqrt{3})N$

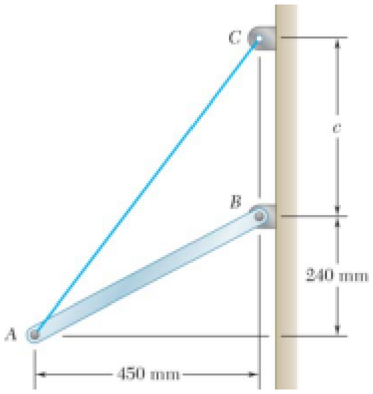
**Exercise:** The reduction gearbox is composed of three axes, and the power is input by axis I, $M_1 = 697Nm$. Gear pitch circle diameter $D_1 = 160mm$, $D_2 = 632mm$, $D_3 = 204mm$, pressure Angle is $20^\circ$, excluding friction, wheel, shaft weight.




Calculate the constrained reaction forces of axes A, B, C and D when rotating at constant speed.

**Ep 18.**

Rod $AB$ is held in place by the cord $AC$. Knowing that the tension in the cord is 1350 N and that $c = 360$ mm, determine the moment about $B$ of the force exerted by the cord at point $A$ by resolving that force into horizontal and vertical components applied (*a*) at point $A$, (*b*) at point $C$.

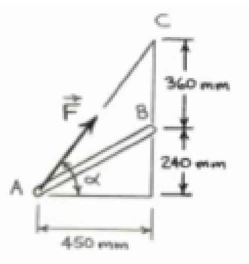



## SOLUTION

**Free-Body Diagram of Rod $AB$:**

(*a*) $F = 1350\text{ N}$ $\qquad AC = \sqrt{(450)^2 + (600)^2} = 750\text{ mm}$

$$\cos\alpha = \frac{450}{750} = 0.6 \qquad \sin\alpha = \frac{600}{750} = 0.8$$



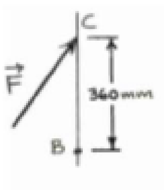

$$\vec{\mathbf{F}} = F\cos\alpha\mathbf{i} + F\sin\alpha\mathbf{j}$$
$$= (1350\text{ N})0.6\mathbf{i} + (1350\text{ N})0.8\mathbf{j}$$
$$= (810\text{ N})\mathbf{i} + (1080\text{N})\mathbf{j}$$

$$\vec{\mathbf{r}}_{A/B} = -(0.45\text{ m})\mathbf{i} - (0.24\text{ m})\mathbf{j}$$

$$\vec{\mathbf{M}}_B = \vec{\mathbf{r}}_{A/B} \times \vec{\mathbf{F}} = (-0.45\mathbf{i} - 0.24\mathbf{j}) \times (810\mathbf{i} + 1080\mathbf{j})$$
$$= -486\mathbf{k} + 194.4\mathbf{k}$$
$$= -(291.6\text{ N}\cdot\text{m})\mathbf{k}$$

$$\mathbf{M}_B = 292\text{ N}\cdot\text{m}$$

(*b*) $\qquad \mathbf{F} = (810\text{ N})\mathbf{i} + (1080\text{ N})\mathbf{j}$




$$\mathbf{r}_{C/B} = (0.36\text{ m})\mathbf{j}$$

$$\mathbf{M}_B = \mathbf{r}_{C/B} \times \mathbf{F} = 0.36\mathbf{j} \times (810\mathbf{i} + 1080\mathbf{j})$$
$$= -(291.6\text{ N}\cdot\text{m})\mathbf{k}$$

$$\mathbf{M}_B = 292\text{ N}\cdot\text{m}$$

**Ep 19.**

Determine the moment about the origin $O$ of the force $\mathbf{F} = 4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ that acts at a Point $A$. Assume that the position vector of $A$ is ($a$) $\mathbf{r} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$, ($b$) $\mathbf{r} = -8\mathbf{i} + 6\mathbf{j} - 10\mathbf{k}$, (c) $\mathbf{r} = 8\mathbf{i} - 6\mathbf{j} + 5\mathbf{k}$.

**SOLUTION**

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$$

($a$)
$$\mathbf{M}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -4 \\ 4 & -3 & 5 \end{vmatrix}$$

$$= (15-12)\mathbf{i} + (-16-10)\mathbf{j} + (-6-12)\mathbf{k} \qquad \mathbf{M}_O = 3\mathbf{i} - 26\mathbf{j} - 18\mathbf{k}$$

($b$)
$$\mathbf{M}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -8 & 6 & -10 \\ 4 & -3 & 5 \end{vmatrix}$$

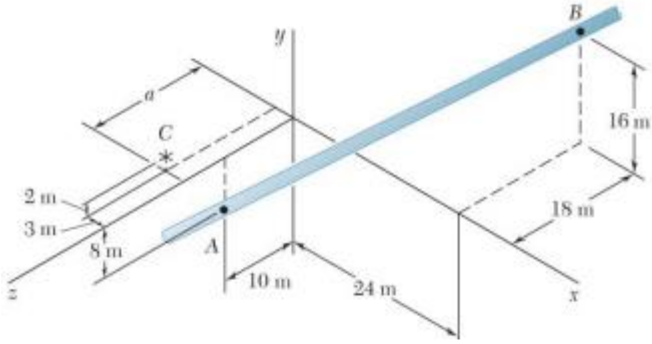
$$= (30-30)\mathbf{i} + (-40+40)\mathbf{j} + (24-24)\mathbf{k} \qquad \mathbf{M}_O = 0$$

($c$)
$$\mathbf{M}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & -6 & 5 \\ 4 & -3 & 5 \end{vmatrix}$$

$$= (-30+15)\mathbf{i} + (20-40)\mathbf{j} + (-24+24)\mathbf{k} \qquad \mathbf{M}_O = -15\mathbf{i} - 20\mathbf{j}$$

**Ep 20.**

Determine the value of $a$ that minimizes the perpendicular distance from Point $C$ to a section of pipeline that passes through Points $A$ and $B$.

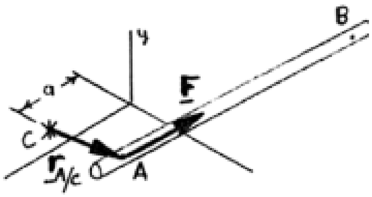



## SOLUTION

Assuming a force **F** acts along $AB$,

$$|\mathbf{M}_C| = |\mathbf{r}_{A/C} \times \mathbf{F}| = F(d)$$

where $d$ = perpendicular distance from $C$ to line $AB$




$$\mathbf{F} = \boldsymbol{\lambda}_{AB} F$$

$$= \frac{(24\text{ m})\mathbf{i} + (24\text{ m})\mathbf{j} - (28\text{ m})\mathbf{k}}{\sqrt{(24)^2 + (24)^2 + (28)^2}\text{ m}} F$$

$$= \frac{F}{11}(6)\mathbf{i} + (6)\mathbf{j} - (7)\mathbf{k}$$

$$\mathbf{r}_{A/C} = (3\text{ m})\mathbf{i} - (10\text{ m})\mathbf{j} - (a - 10\text{ m})\mathbf{k}$$

$$\mathbf{M}_C = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -10 & 10a \\ 6 & 6 & -7 \end{vmatrix} \frac{F}{11}$$

$$= [(10 + 6a)\mathbf{i} + (81 - 6a)\mathbf{j} + 78\,\mathbf{k}]\frac{F}{11}$$

Since $\quad |\mathbf{M}_C| = \sqrt{|\mathbf{r}_{A/C} \times \mathbf{F}^2|} \quad$ or $\quad |\mathbf{r}_{A/C} \times \mathbf{F}^2| = (dF)^2$

$$\frac{1}{121}(10 + 6a)^2 + (81 - 6a)^2 + (78)^2 = d^2$$

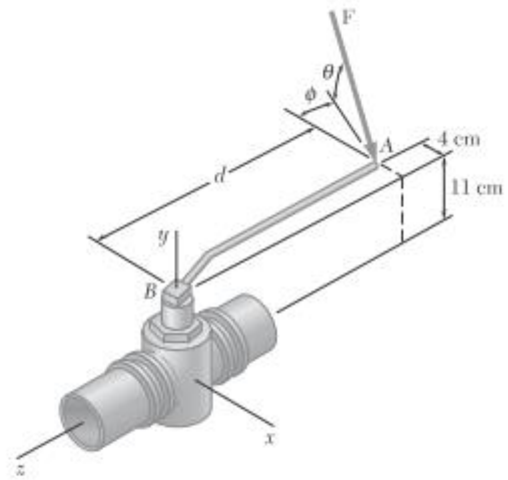
Setting $\frac{d}{da}(d^2) = 0$ to find $a$ to minimize $d$:

$$\frac{1}{121}[2(6)(10 + 6a) + 2(-6)(81 - 6a)] = 0$$

Solving $\quad a = 5.92\text{ m} \qquad$ or $\quad a = 5.92\text{ m}$

**Ep 21.**

To loosen a frozen valve, a force **F** of magnitude 70 N is applied to the handle of the valve. Knowing that $\theta = 25°$, $M_x = -7.32$ N·m, and $M_z = -5.16$ N·m, determine $\phi$ and $d$.




**SOLUTION**

We have $\quad \Sigma \mathbf{M}_O: \quad \mathbf{r}_{A/O} \times \mathbf{F} = \mathbf{M}_O$

where $\quad \mathbf{r}_{A/O} = -(4 \text{ cm})\mathbf{i} + (11 \text{ cm})\mathbf{j} - (d)\mathbf{k}$

$$\mathbf{F} = F(\cos\theta\cos\phi\mathbf{i} - \sin\theta\mathbf{j} + \cos\theta\sin\phi\mathbf{k})$$

For $\quad F = 70 \text{ N}, \quad \theta = 25°$

$$\mathbf{F} = (70 \text{ N})[(0.90631\cos\phi)\mathbf{i} - 0.42262\mathbf{j} + (0.90631\sin\phi)\mathbf{k}]$$

$$\mathbf{M}_O = (70 \text{ N})\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 11 & -d \\ -0.90631\cos\phi & -0.42262 & 0.90631\sin\phi \end{vmatrix} \text{cm}$$

$$= (70 \text{ N})[(9.9694\sin\phi - 0.42262d)\mathbf{i} + (-0.90631d\cos\phi + 3.6252\sin\phi)\mathbf{j} + (1.69048 - 9.9694\cos\phi)\mathbf{k}] \text{ cm}$$

and

$$M_x = (70 \text{ N})(9.9694\sin\phi - 0.42262d)\text{cm} = 732 \text{ N·cm} \qquad (1)$$

$$M_y = (70 \text{ N})(-0.90631d\cos\phi + 3.6252\sin\phi) \text{ cm} \qquad (2)$$

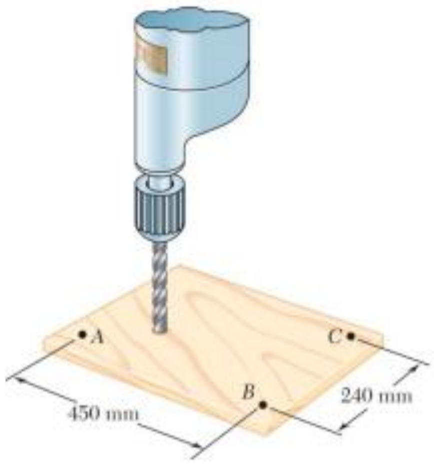
$$M_z = (70 \text{ N})(1.69048 - 9.9694\cos\phi) \text{ cm} = -516 \text{ N·cm} \qquad (3)$$

From Equation (3) $\quad \phi = \cos^{-1}\left(\dfrac{634.33}{697.86}\right) = 24.636° \qquad$ or $\quad \phi = 24.6°$

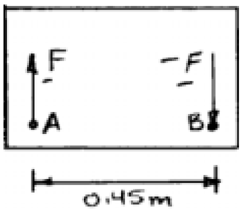
From Equation (1) $\quad d = \left(\dfrac{1022.90}{29.583}\right) = 34.577 \text{ cm} \qquad$ or $\quad d = 0.346 \text{ m}$

**Ep 22.**

A piece of plywood in which several holes are being drilled successively has been secured to a workbench by means of two nails. Knowing that the drill exerts a 12-N·m couple on the piece of plywood, determine the magnitude of the resulting forces applied to the nails if they are located (*a*) at *A* and *B*, (*b*) at *B* and *C*, (*c*) at *A* and *C*.



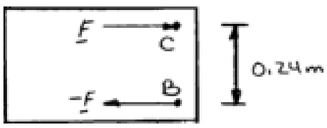

**SOLUTION**




(*a*) $M = Fd$

$12\ \text{N}\cdot\text{m} = F(0.45\ \text{m})$

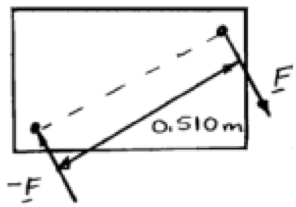
$F = 26.7\ \text{N}$




(*b*) $M = Fd$

$12\ \text{N}\cdot\text{m} = F(0.24\ \text{m})$

$F = 50.0\ \text{N}$




(*c*) $M = Fd \qquad d = \sqrt{(0.45\ \text{m})^2 + (0.24\ \text{m})^2}$

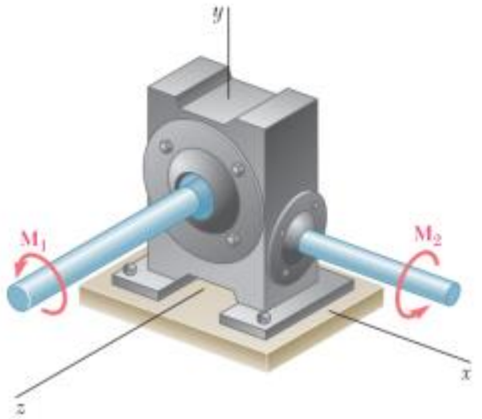
$= 0.510\ \text{m}$

$12\ \text{N}\cdot\text{m} = F(0.510\ \text{m})$

$F = 23.5\ \text{N}$

**Ep 23.**

The two shafts of a speed-reducer unit are subjected to couples of magnitude $M_1$ = 15 N·m and $M_2$ = 3 N·m, respectively. Replace the two couples with a single equivalent couple, specifying its magnitude and the direction of its axis.




**SOLUTION**

$$M_1 = (15\ \text{N}\cdot\text{m})\mathbf{k}$$

$$M_2 = (3\ \text{N}\cdot\text{m})\mathbf{i}$$

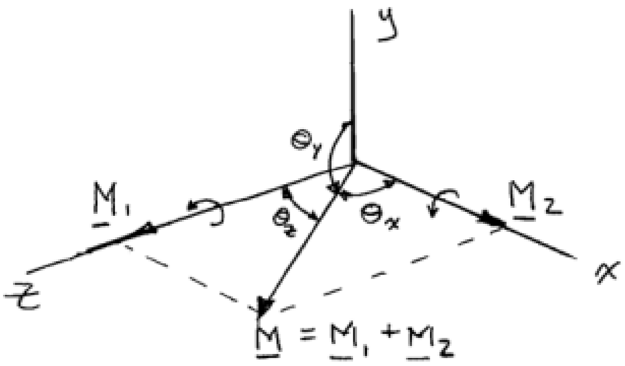
$$M = \sqrt{M_1^2 + M_2^2}$$
$$= \sqrt{(15)^2 + (3)^2}$$
$$= 15.30\ \text{N}\cdot\text{m}$$

$$\tan\theta_x = \frac{15}{3} = 5$$

$$\theta_x = 78.7°$$

$$\theta_y = 90°$$

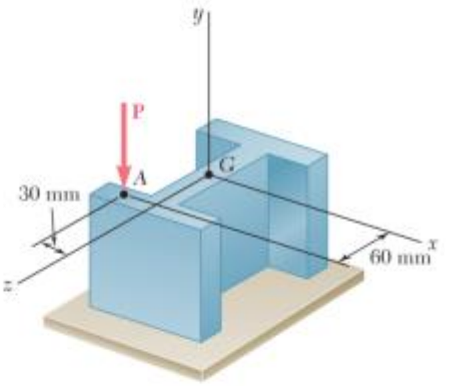
$$\theta_z = 90° - 78.7°$$
$$= 11.30°$$



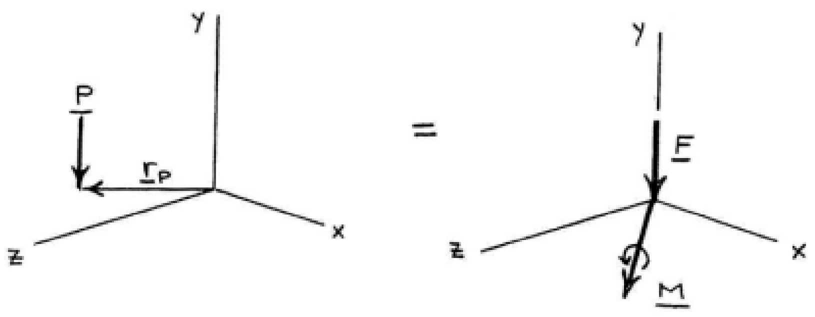

$M = 15.30$ N·m; $\theta_x$ = 78.7°, $\theta_y$ = 90.0°, $\theta_z$ = 11.30°

**Ep 24.**

An eccentric, compressive 250-kN force **P** is applied to the end of a column. Replace **P** with an equivalent force-couple system at *G*.




**SOLUTION**




Have $\qquad \Sigma\mathbf{F}: \qquad -(250 \text{ kN})\mathbf{j} = \mathbf{F}$

or $\mathbf{F} = -(250 \text{ kN})\mathbf{j}$

Also have $\qquad \Sigma\mathbf{M}_G: \qquad \mathbf{r}_P \times \mathbf{P} = \mathbf{M}$

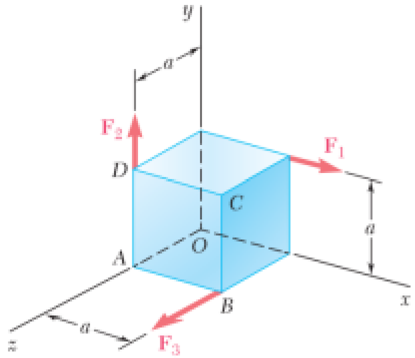
$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -0.030 & 0 & 0.060 \\ 0 & -250 & 0 \end{vmatrix} \text{kN}\cdot\text{m} = \mathbf{M}$$

$$\therefore \; \mathbf{M} = (15 \text{ kN}\cdot\text{m})\mathbf{i} + (7.5 \text{ kN}\cdot\text{m})\mathbf{k}$$

or $\mathbf{M} = (15.00 \text{ kN}\cdot\text{m})\mathbf{i} + (7.50 \text{ kN}\cdot\text{m})\mathbf{k}$

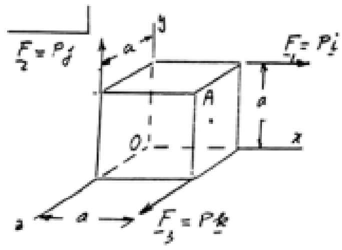
**Ep 25.**

Three forces of the same magnitude $P$ act on a cube of side $a$ as shown. Replace the three forces by an equivalent wrench and determine ($a$) the magnitude and direction of the resultant force **R**, ($b$) the pitch of the wrench, ($c$) the axis of the wrench.




## SOLUTION

Force-couple system at $O$:

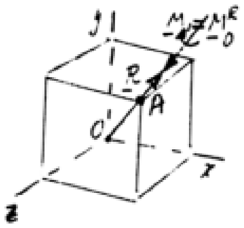


$$\mathbf{R} = P\mathbf{i} + P\mathbf{j} + P\mathbf{k} = P(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

$$\mathbf{M}_O^R = a\mathbf{j} \times P\mathbf{i} + a\mathbf{k} \times P\mathbf{j} + a\mathbf{i} \times P\mathbf{k}$$
$$= -Pa\mathbf{k} - Pa\mathbf{i} - Pa\mathbf{j}$$

$$\mathbf{M}_O^R = -Pa(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

Since **R** and $\mathbf{M}_O^R$ have the same direction, they form a wrench with $\mathbf{M}_1 = \mathbf{M}_O^R$. Thus, the axis of the wrench is the diagonal $OA$. We note that



$$\cos\theta_x = \cos\theta_y = \cos\theta_z = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$R = P\sqrt{3} \quad \theta_x = \theta_y = \theta_z = 54.7°$$
$$M_1 = M_O^R = -Pa\sqrt{3}$$

$$\text{Pitch} = p = \frac{M_1}{R} = \frac{-Pa\sqrt{3}}{P\sqrt{3}} = -a$$

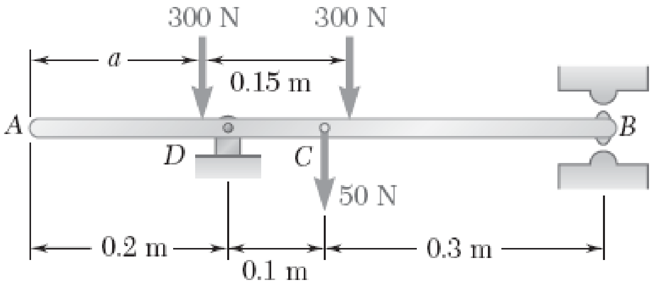
($a$) $R = P\sqrt{3} \quad \theta_x = \theta_y = \theta_z = 54.7°$

($b$) $-a$

($c$) Axis of the wrench is diagonal $OA$.

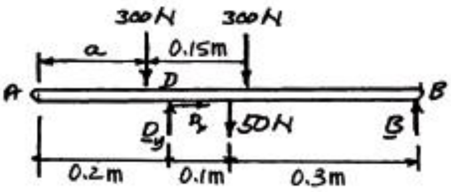
**Ep 26.**

For the beam and loading shown, determine the range of the distance $a$ for which the reaction at $B$ does not exceed 100 N downward or 200 N upward.

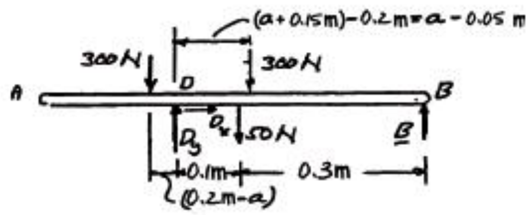



## SOLUTION

Assume $B$ is positive when directed ↑




Sketch showing distance from $D$ to forces.




$$+\circlearrowleft \Sigma M_D = 0: \quad (300\text{ N})(0.2\text{ m} - a) - (300\text{ N})(a - 0.05\text{ m}) - (50\text{ N})(0.1\text{ m}) + 0.4B = 0$$

$$-600a + 70 + 0.4B = 0$$

$$a = \frac{(70 + 0.4B)}{600} \qquad (1)$$

For $B = 100\text{ N}\downarrow = -100\text{ N}$, Eq. (1) yields:

For $B = 100\text{ N}\downarrow = -100\text{ N}$, Eq. (1) yields:

$$a \geq \frac{[70 + 0.4(-100)]}{600} = \frac{30}{600} = 0.05\text{ m} \qquad a \geq 50\text{ mm}$$

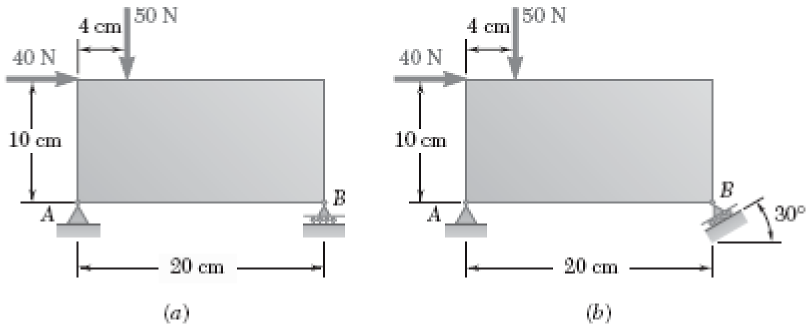
For $B = 200\text{ N}\uparrow = +200\text{ N}$, Eq. (1) yields:

$$a \leq \frac{[70 + 0.4(200)]}{600} = \frac{150}{600} = 0.25\text{ m} \qquad a \leq 250\text{ mm}$$

Required range: $\quad 0.05\text{ m} \leq a \leq 0.25\text{ m}$

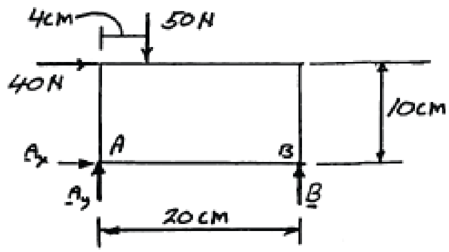
**Ep 27.**

For each of the plates and loadings shown, determine the reactions at $A$ and $B$.




**SOLUTION**

($a$) **Free-Body Diagram:**




$+\circlearrowleft \Sigma M_A = 0$: $B(20\text{ cm}) - (50\text{ N})(4\text{ cm}) - (40\text{ N})(10\text{ cm}) = 0$

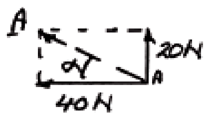
$B = +30\text{ N}$ $\quad\quad$ $\mathbf{B} = 30.0\text{ N}\uparrow$

$\xrightarrow{+} \Sigma F_x = 0$: $A_x + 40\text{ N} = 0$

$A_x = -40\text{ N}$ $\quad\quad$ $\mathbf{A}_x = 40.0\text{ N} \longleftarrow$

$+\uparrow \Sigma F_y = 0$: $A_y + B - 50\text{ N} = 0$

$A_y + 30\text{ N} - 50\text{ N} = 0$

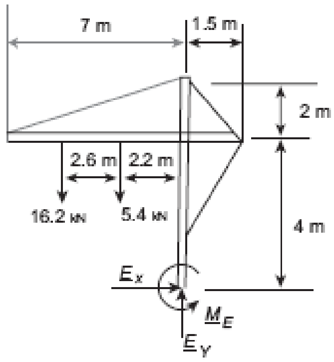



$A_y = +20\text{ N}$ $\quad\quad$ $\mathbf{A}_y = 20.0\text{ N}\uparrow$

$\alpha = 26.56°$

$A = 44.72\text{ N}$ $\quad\quad$ $\mathbf{A} = 44.7\text{ N} \searrow 26.6°$

(*b*) **Free-Body Diagram:**




$\xrightarrow{+} \Sigma F_x = 0$: $\quad E_x = 0$

$+\uparrow \Sigma F_y = 0$: $\quad E_y - 16.2 \text{ kN} - 5.4 \text{ kN} = 0$

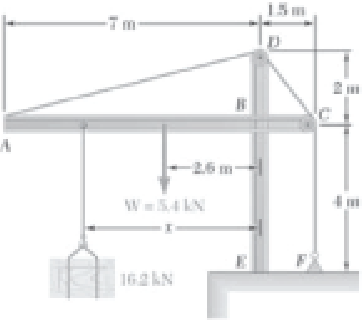
$E_y = 21.6 \text{ kN}$ $\qquad$ or $\mathbf{E} = 21.6 \text{ kN} \uparrow$

$+\circlearrowleft \Sigma M_E = 0$: $\quad M_E + (16.2 \text{ kN})(4.8 \text{ m}) + (5.4 \text{ kN})(2.6 \text{ m}) = 0$

$M_E = -91.8 \text{ kN} \cdot \text{m}$ $\qquad$ or $\mathbf{M}_E = 91.8 \text{ kN} \cdot \text{m} \circlearrowright$

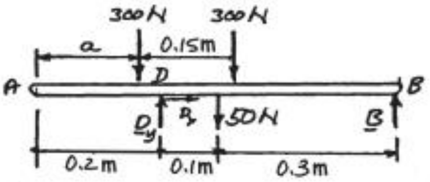
**Ep 28.**

For the rig and crate of Prob. 4.43, assuming that the cable is anchored at $F$ as shown, determine (*a*) the required tension in cable $ADCF$ if the maximum value of the couple at $E$ is to be as small as possible as $x$ varies from 0.6 m to 7 m, (*b*) the corresponding maximum value of the couple.

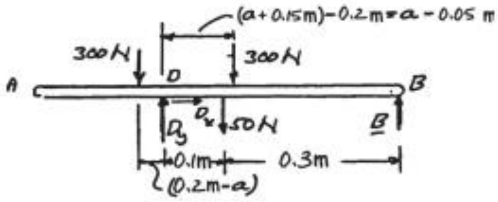



## SOLUTION

Assume $B$ is positive when directed $\uparrow$




Sketch showing distance from $D$ to forces.




$$+\circlearrowleft \Sigma M_D = 0: \quad (300 \text{ N})(0.2 \text{ m} - a) - (300 \text{ N})(a - 0.05 \text{ m}) - (50 \text{ N})(0.1 \text{ m}) + 0.4B = 0$$

$$-600a + 70 + 0.4B = 0$$

$$a = \frac{(70 + 0.4B)}{600} \qquad (1)$$

For $B = 100 \text{ N}\downarrow = -100 \text{ N}$, Eq. (1) yields:

$$a \geq \frac{[70 + 0.4(-100)]}{600} = \frac{30}{600} = 0.05 \text{ m} \qquad a \geq 50 \text{ mm}$$

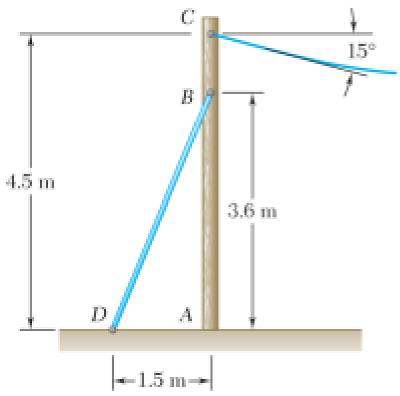
For $B = 200 \text{ N}\uparrow = +200 \text{ N}$, Eq. (1) yields:

$$a \leq \frac{[70 + 0.4(200)]}{600} = \frac{150}{600} = 0.25 \text{ m} \qquad a \leq 250 \text{ mm}$$

Required range: $\qquad 0.05 \text{ m} \leq a \leq 0.25 \text{ m}$

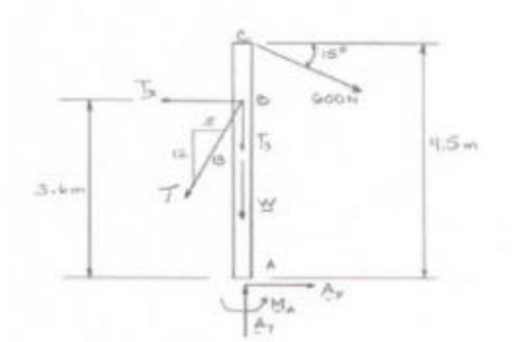
**Ep 29.**

A 175-kg utility pole is used to support at $C$ the end of an electric wire. The tension in the wire is 600 N, and the wire forms an angle of 15° with the horizontal at $C$. Determine the largest and smallest allowable tensions in the guy cable $BD$ if the magnitude of the couple at $A$ may not exceed 500 N·m.




## SOLUTION

**Free-Body Diagram:**



*Geometry*:

Distance $BD = \sqrt{(1.5)^2 + (3.6)^2} = 3.90$ m

Note also that: $W = mg = (175 \text{ kg})(9.81 \text{ m/s}^2) = 1716.75$ N

With $\mathbf{M}_A = 500$ N·m clockwise: (i.e. corresponding to $T_{\max}$)

$$+\circlearrowright \Sigma M_A = 0: \qquad -500 \text{ N}\cdot\text{m} - [(600 \text{ N})\cos 15^\circ](4.5 \text{ m}) + \left[\left(\frac{1.5}{3.90}\right)T_{\max}\right](3.6 \text{ m}) = 0$$

$T_{\max} = 2244.7$ N $\qquad$ or $T_{\max} = 2240$ N

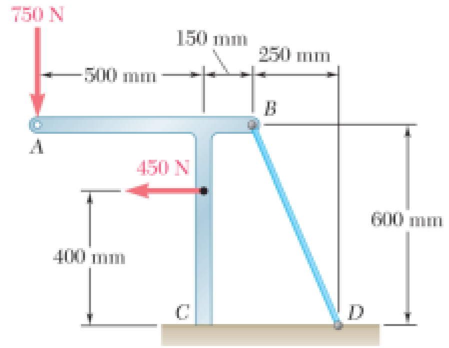
With $\mathbf{M}_A = 500$ N·m counter-clockwise: (i.e. corresponding to $T_{\min}$)

$$+\circlearrowright \Sigma M_A = 0: \qquad 500 \text{ N}\cdot\text{m} - [(600 \text{ N})\cos 15^\circ](4.5 \text{ m}) + \left[\left(\frac{1.5}{3.90}\right)T_{\min}\right](3.6 \text{ m}) = 0$$

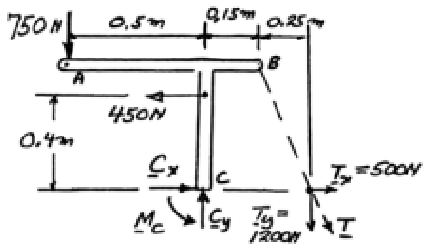
$T_{\min} = 1522.44$ N $\qquad$ or $T_{\min} = 1522$ N

**Ep 29.**

Knowing that the tension in wire *BD* is 1300 N, determine the reaction at the fixed support *C* of the frame shown.




**SOLUTION**




$$T = 1300 \text{ N}$$

$$T_x = \frac{5}{13}T$$

$$= 500 \text{ N}$$

$$T_y = \frac{12}{13}T$$

$$= 1200 \text{ N}$$

$\xrightarrow{+} \Sigma M_x = 0$: $C_x - 450 \text{ N} + 500 \text{ N} = 0$ $\quad C_x = -50 \text{ N}$ $\qquad \mathbf{C}_x = 50 \text{ N} \leftarrow$

$+\uparrow \Sigma F_y = 0$: $C_y - 750 \text{ N} - 1200 \text{ N} = 0$ $\quad C_y = +1950 \text{ N}$

$\mathbf{C}_y = 1950 \text{ N} \uparrow$

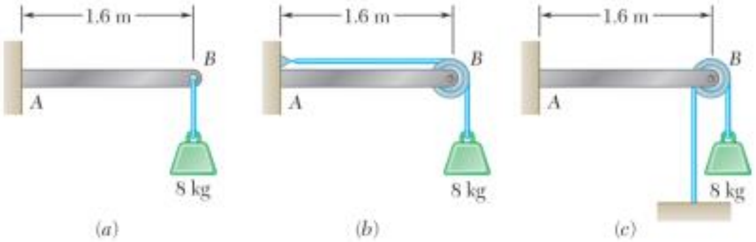

$\mathbf{C} = 1951 \text{ N} \measuredangle 88.5°$

$+\circlearrowleft \Sigma M_C = 0$: $M_C + (750 \text{ N})(0.5 \text{ m}) + (4.50 \text{ N})(0.4 \text{ m})$
$- (1200 \text{ N})(0.4 \text{ m}) = 0$

$M_C = -75.0 \text{ N} \cdot \text{m}$ $\qquad \mathbf{M}_C = 75.0 \text{ N} \cdot \text{m} \circlearrowright$

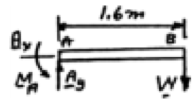
**Ep 30.**

An 8-kg mass can be supported in the three different ways shown. Knowing that the pulleys have a 100-mm radius, determine the reaction at $A$ in each case.




**SOLUTION**

$$W = mg = (8\text{ kg})(9.81\text{ m/s}^2) = 78.480\text{ N}$$



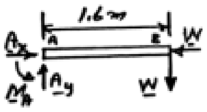

(*a*) $\Sigma F_x = 0:\quad A_x = 0$

$+\uparrow \Sigma F_y = 0:\quad A_y - W = 0 \qquad \mathbf{A}_y = 78.480\text{ N}\uparrow$

$+\circlearrowleft \Sigma M_A = 0:\quad M_A - W(1.6\text{ m}) = 0$

$M_A = +(78.480\text{ N})(1.6\text{ m}) \qquad \mathbf{M}_A = 125.568\text{ N}\cdot\text{m}\circlearrowleft$

$\mathbf{A} = 78.5\text{ N}\uparrow,\quad \mathbf{M}_A = 125.6\text{ N}\cdot\text{m}\circlearrowleft$




(*b*) $\xrightarrow{+} \Sigma F_x = 0:\quad A_x - W = 0 \qquad \mathbf{A}_x = 78.480\uparrow$

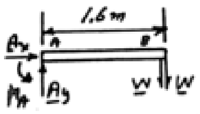
$+\uparrow \Sigma F_y = 0:\quad A_y - W = 0 \qquad \mathbf{A}_y = 78.480 \leftarrow$

$$\mathbf{A} = (78.480\text{ N})\sqrt{2} = 110.987\text{ N} \measuredangle 45^\circ$$

$+\circlearrowleft \Sigma M_A = 0:\quad M_A - W(1.6\text{ m}) = 0$

$M_A = +(78.480\text{ N})(1.6\text{ m}) \qquad \mathbf{M}_A = 125.568\text{ N}\cdot\text{m}\circlearrowleft$

$\mathbf{A} = 111.0\text{ N} \measuredangle 45^\circ,\quad \mathbf{M}_A = 125.6\text{ N}\cdot\text{m}\circlearrowleft$




(*c*) $\Sigma F_x = 0:\quad A_x = 0$

$+\uparrow \Sigma F_y = 0:\quad A_y - 2W = 0$

$$A_y = 2W = 2(78.480\text{ N}) = 156.960\text{ N}\uparrow$$

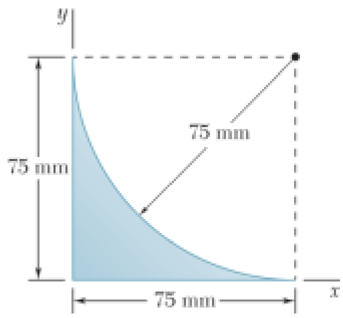
$+\circlearrowleft \Sigma M_A = 0:\quad M_A - 2W(1.6\text{ m}) = 0$

$M_A = +2(78.480\text{ N})(1.6\text{ m}) \qquad \mathbf{M}_A = 251.14\text{ N}\cdot\text{m}\circlearrowleft$

$\mathbf{A} = 157.0\text{ N}\uparrow,\quad \mathbf{M}_A = 251\text{ N}\cdot\text{m}\circlearrowleft$

**Ep 31.**

Locate the centroid of the plane area shown.




**SOLUTION**

Area 1: Square 75 mm by 75 mm.

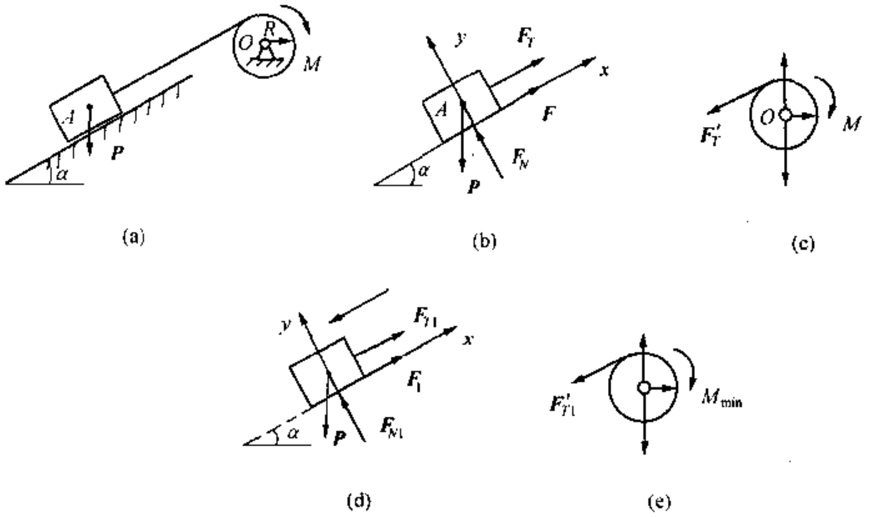
Area 2: Quarter circle radius of 75 mm.

| | $A$, mm$^2$ | $\bar{x}$, mm | $\bar{y}$, mm | $\bar{x}A$, mm$^3$ | $\bar{y}A$, mm$^3$ |
|---|---|---|---|---|---|
| 1 | $75 \times 75 = 5625$ | 37.5 | 37.5 | 210,938 | 210,938 |
| 2 | $-\frac{\pi}{4} \times 75^2 = -4417.9$ | 43.169 | 43.169 | -190,715 | −190,715 |
| Σ | 1207.10 | | | 20,223 | 20,223 |

Then by symmetry $\quad \bar{X} = \bar{Y} = \dfrac{\Sigma \bar{x} A}{\Sigma A}$

$\bar{X}(1207.14 \text{ mm}^2) = 20,223 \text{ mm}^3 \qquad \bar{X} = \bar{Y} = 16.75 \text{ mm}$

**Ep 32.** As shown in Ep 32 (a), block A with a weight of P=1000N is placed on the slope with an inclination of $\alpha = 30^{\circ}$. The static sliding friction coefficient $f_s = 0.2$ between block A and the inclined plane, ignore the weight of the wire rope. The wheel is a homogeneous wheel with a radius of R=0.1m.（1）When the couple moments of force applied to the pulley are $\mathrm{M}_1 = 40\mathrm{Nm}$ and $\mathrm{M}_2 = 60\mathrm{Nm}$, the system is at rest, and calculate the friction force between block A and the inclined plane.（2）The range of the couple moment M when the system is in equilibrium.



Ep 10

**SOLUTION:**

（1）Take block A and pulley respectively, and the force diagram is shown in Figure (b) (c).

For block A, there are: $\sum \mathrm{X} = 0$ $\quad F_T + F - P\sin\alpha = 0$ （1）

For pulleys, there are: $\sum \mathrm{m}_o(\vec{F}) = 0$ $\quad F_T' R - M = 0$ （2）

We can get $$F_T' = \frac{M}{R}$$

When $M = M_1 = 40Nm$, substitute $F_T' = 400N$ into（1）, we get $F_1 = 100N$（go up along the slope）

When $M = M_2 = 60Nm$, substitute $F_T' = 600N$ into（1）, we get $F_1 = -100N$（go down along the slope）

At this time, $F_N = P\cos\alpha = 500\sqrt{3}N$ $\quad F_{\max} = f_s F_N = 100\sqrt{3}N$

（2）Block A and pulley are respectively taken. When the couple moment M added to the pulley is $M_{\min}$, block A is in the critical state of sliding down, and its force diagrams are shown in Figure (d) (e).

For block A, from

$$\sum X = 0 \qquad F_{T1} + F_1 - P\sin\alpha = 0$$

$$\sum Y = 0 \qquad F_{N1} - P\cos\alpha = 0$$

We get $\qquad F_1 = f_s F_{N1}$

Solve it $\qquad F_{T1} = P\sin\alpha - f_s F_{N1} = P\sin\alpha - f_s P\cos\alpha = 326.8N$

On the pulley, from

$$\sum m_o(\vec{F}) = 0 \qquad F'_{T1}R - M_{\min} = 0$$

We get $\qquad M_{\min} = 32.68Nm$

In the same way, when $M = M_{\max}$, block A is in the critical state of moving up, and the frictional force is moving down, there are:

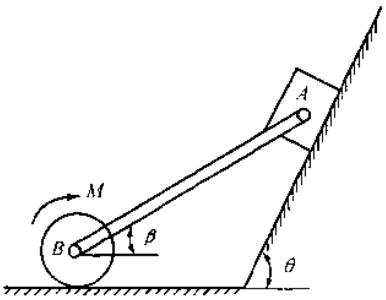
$$F_{T2} = P\sin\alpha + f_s F_{N1} = P\sin\alpha + f_s P\cos\alpha = 673.2N$$

$$M_{max} = 67.32Nm$$

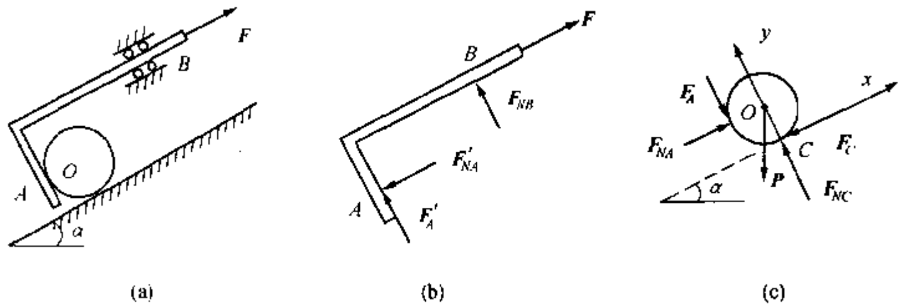
Therefore, when the system is in equilibrium, the value range of the moment M of the force couple is

$$32.68Nm \le M \le 67.32Nm$$

**Exercise:** As shown in the figure on the right, block A weighs $P_1 = 100N$, homogeneous wheel B weighs $P_2 = 300N$, $R = 0.2m$, the dead weight of rod AB is not counted, the static sliding friction coefficient between wheel and horizontal plane is $f_s = 0.4$, the rolling friction is not counted, and the bevel is smooth. In order to balance the system in the figure ($\beta = 30^\circ$), what is the couple M of torque applied on wheel B? What is the sliding friction between the wheel and the horizontal plane?



**Ep 33.** As shown in Ep 33 (a), homogeneous cylinder O weights P, the radius is R, on the slope Angle is $\alpha = 30^\circ$, a no self-respecting right-angle bend rod AB block, cylindrical and bevel and static sliding friction factor between the bending pole is $f_s$, ignore the rolling friction and B is smooth. Find the minimum force $F_{\min}$ to pull the bending rod, and force F is parallel to the inclined plane.




Ep 33

**SOLUTION:**

(1) The right-angle bending bar and cylinder are respectively studied. Assuming that the cylinder has A pure upward sliding trend, the friction force at C is downward, but the relative motion trend at A is not obvious. The direction of friction is assumed as shown in the figure, and the force diagram is shown in (b) and (C) respectively.

For the cylindrical

$$\sum X = 0 \qquad F_{NA} - F_C - P\sin\alpha = 0 \qquad (1)$$

$$\sum Y = 0 \qquad F_{NC} - F_A - P\cos\alpha = 0 \qquad (2)$$

$$\sum m_o(\vec{F}) = 0 \qquad F_A R - F_C R = 0 \qquad (3)$$

Because the critical sliding state has been reached at C,

$$F_C = f_s F_{NC} \qquad (4)$$

Solve it simultaneously, and get $F_{NA} = \dfrac{P\sin\alpha + f_s P(\cos\alpha - \sin\alpha)}{1 - f_s}$

For the equilibrium equation is arranged for the bent bar

$$\sum X = 0 \qquad F - F'_{NA} = 0$$

We get

$$F = \frac{P\sin\alpha + f_s P(\cos\alpha - \sin\alpha)}{1 - f_s} \qquad (A)$$

(2) The right-angle bending bar and cylinder are respectively studied. Suppose the cylinder is in an upward pure rolling trend, the friction force at C is downward, the friction force at A is still assumed as shown in the figure, and the force is still shown as (b) and (C).

For the cylinder, there are still equations (1), (2) and (3), but (4) is not true.

For the wheel, the critical sliding state should be reached at A, and there is

$$F_A = f_s F_{NA} \qquad (4')$$

Equations (1), (2), (3) and (4') can be solved simultaneously

$$F_{NA} = \frac{P\sin\alpha}{1-f_s}$$

The equilibrium equation is arranged for the bent bar

$$\sum X = 0 \qquad F - F'_{NA} = 0$$

Solve it

$$F = \frac{P\sin\alpha}{1-f_s} \qquad \text{(B)}$$

Comparing (A) and (B), because of $\alpha = 30^\circ$, $\cos\alpha - \sin\alpha > 0$, we have

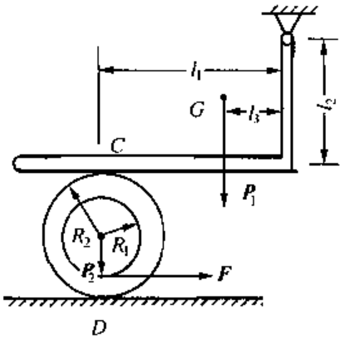
$$\frac{P\sin\alpha + f_s P(\cos\alpha - \sin\alpha)}{1-f_s} > \frac{P\sin\alpha}{1-f_s}$$

Therefore, under the trend of pure sliding and pure rolling of cylinder, the force F required for pure rolling is small.

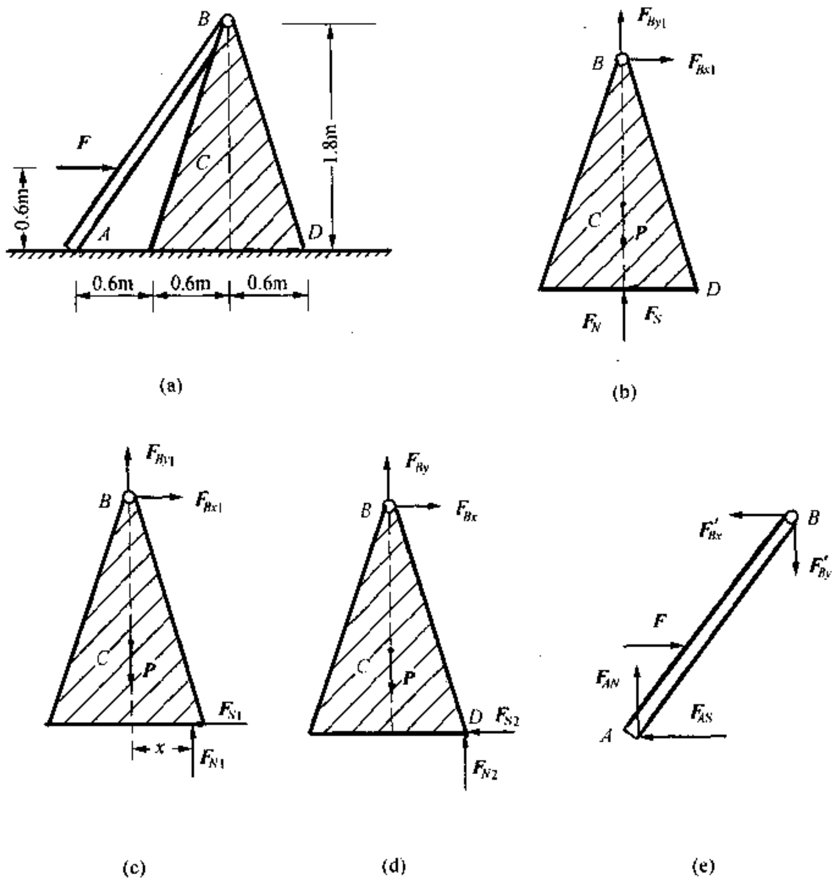
(3) If the cylinder is in the trend of rolling and sliding, the required F force must be greater than that required for pure rolling. Therefore, the minimum force required to pull the bending rod is

$$F_{\min} = \frac{P}{2(1-f_s)}$$

**Exercise:** In the system shown in the figure on the right, the weight of the right-angle bending bar is $P_1$, the center of gravity is at point G, the wheel shaft weight is $P_2$, $P_1 = 2P_2$, and the static sliding friction coefficient at C and D is $f_s = 0.4$. The dimensions are shown in the figure, $l_1 = 2l_2 = 3l_3$, $R_2 = 2R_1$, excluding the rolling friction. Find the maximum horizontal tension F at equilibrium.



**Ep 33.** As shown in Figure 12 (a), the weightless bar AB and the weight P are homogenous tri-prism C hinged at Point B, and there is a horizontal right force F acting on the bar AB. The static sliding friction coefficient $f_s = 0.4$ between the bar AB and the horizontal plane, as well as between the prism and the horizontal plane, is shown in the figure. Calculate the maximum value of the force F that keeps the system in balance.



Ep 34

**SOLUTION:**

If the force diagram (b) is applied, the moment at point B will definitely tip over. So the correct force diagram should be (c).

（1） When the triangular prism slides to the right, its force is shown in Figure (c). At this time there are:

$$F_{s1} = f_s F_{N1}$$

From $\sum m_B(\vec{F}) = 0$ $\qquad xF_{N1} - 1.8F_{s1} = 0$

That is $\qquad xF_{N1} - 1.8 f_s F_{N1} = 0$

Solve it $x = 1.8 f_s = 0.72m$

This result indicates that if the prism is in the critical sliding state, the action line of $F_{N1}$ should be 0.72m away from the center of the prism, which is impossible, so the prism will not slip.

（2） When the triangular prism has a trend of turning over around the edges D, its force diagram is shown in Figure (D). From

$$\sum m_D(\vec{F}) = 0 \qquad 0.6P - 1.8F_{Bx} - 0.6F_{By} = 0 \tag{1}$$

Take rod AB and its force diagram is shown in Figure (e). From

$$\sum X = 0 \qquad F - F'_{Bx} - F_{AS} = 0 \tag{2}$$

$$\sum Y = 0 \qquad F_{AN} - F'_{By} = 0 \tag{3}$$

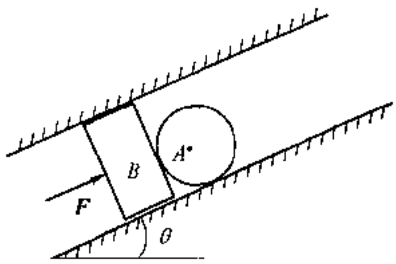
$$\sum m_A(\vec{F}) = 0 \qquad 1.8F'_{Bx} - 1.2F'_{By} - 0.6F = 0 \tag{4}$$

$$F_{AS} = f_s F_{AN} \tag{5}$$

The equations (1-5) can be solved simultaneously, so that the maximum value of the equilibrium force F of the system is

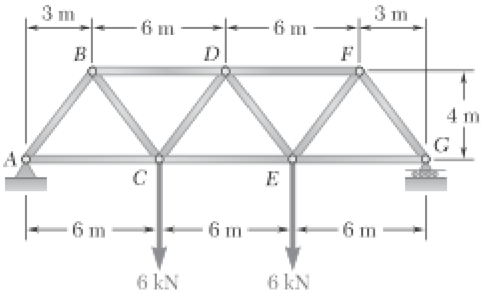
$$F_{\max} = \frac{8}{23}P$$

**Exercise:** In the figure on the right, homogenizer wheel A weighs P, and block B weighs nothing. The static sliding friction coefficient between wheel A and the inclined plane and block B is $f_s$. The contact point between block B and the inclined plane is smooth, and the force F is parallel to the inclined plane. When $\theta = 30^\circ, 45^\circ, 60^\circ$, find the force F to keep the system in balance.

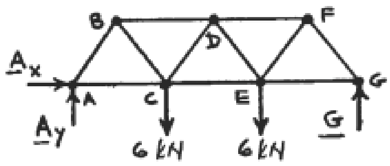


**Ep 33.**

Determine the force in each member of the Warren bridge truss shown. State whether each member is in tension or compression.




## SOLUTION



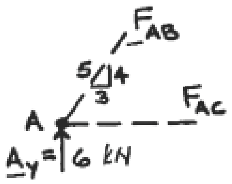

Free body: Truss $\Sigma F_x = 0$: $A_x = 0$

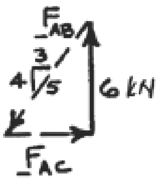
Due to symmetry of truss and loading

$$A_y = G = \frac{1}{2}\text{Total load} = 6\text{ kN} \uparrow$$

Free body: Joint $A$: $\dfrac{F_{AB}}{5} = \dfrac{F_{AC}}{3} = \dfrac{6\text{ kN}}{4}$ $F_{AB} = 7.50\text{ kN}$ $C$

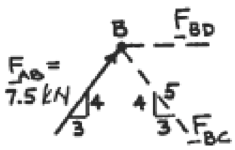
$F_{AC} = 4.50\text{ kN}$ $T$

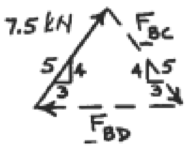


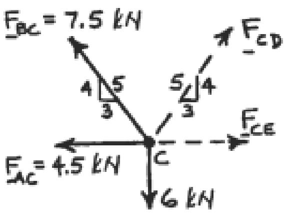

Free body: Joint $B$: $\dfrac{F_{BC}}{5} = \dfrac{F_{BD}}{6} = \dfrac{7.5\text{ kN}}{5}$ $F_{BC} = 7.50\text{ kN}$ $T$

$F_{BD} = 9.00\text{ kN}$ $C$




Free body: Joint $C$:




$$+\uparrow \Sigma F_y = 0: \quad \frac{4}{5}(7.5) + \frac{4}{5}F_{CD} - 6 = 0$$

$F_{CD} = 0$

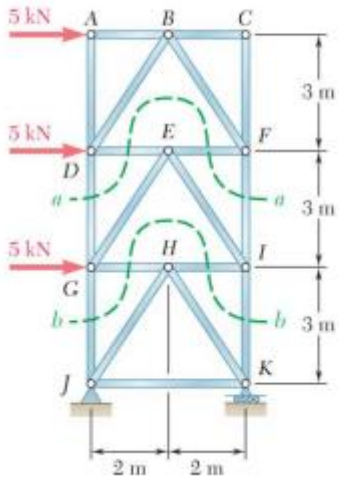
$$\xrightarrow{+} \Sigma F_x = 0: \quad F_{CE} - 4.5 - \frac{3}{5}(7.5) = 0$$

$$+\uparrow F_{CE} = +9\text{ kN}$$

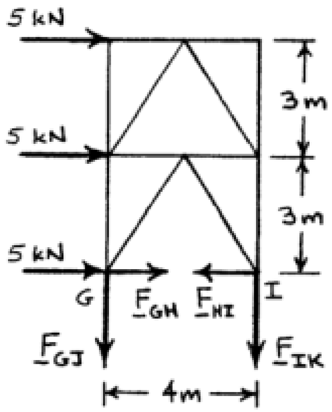
$F_{CE} = 9.00\text{ kN}$ $T$

**Ep 34.**

Determine the force in members *GJ* and *IK* of the truss shown. (*Hint:* Use section *bb*.)




**SOLUTION**




$$+\circlearrowleft \Sigma M_I = 0: \quad F_{GJ}(4\text{ m}) - (5\text{ kN})(6\text{ m}) - (5\text{ kN})(3\text{ m}) = 0$$

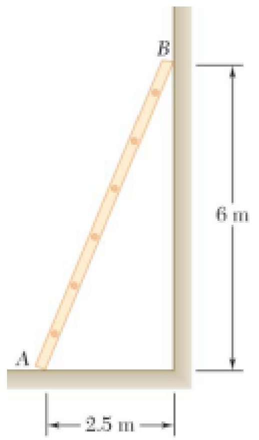
$$F_{GJ} = +11.25\text{ kN} \qquad\qquad F_{GJ} = 11.25\text{ kN} \quad T$$

$$+\uparrow \Sigma F_y = 0: \quad -11.25\text{ kN} - F_{IK} = 0$$

$$F_{IK} = -11.25\text{ kN} \quad F_{IK} = 11.25\text{ kN} \quad C$$

**Ep 35.**

A 6.5-m ladder *AB* leans against a wall as shown. Assuming that the coefficient of static friction $\mu_s$ is the same at *A* and *B*, determine the smallest value of $\mu_s$ for which equilibrium is maintained.

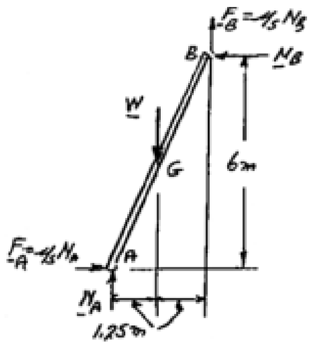



**SOLUTION**

<u>Free body: Ladder</u>

**Motion impending:**

$$F_A = \mu_s N_A$$
$$F_B = \mu_s N_B$$




$$+\circlearrowleft \Sigma M_A = 0: \quad W(1.25 \text{ m}) - N_B(6 \text{ m}) - \mu_s N_B(2.5 \text{ m}) = 0$$

$$N_B = \frac{1.25W}{6 + 2.5\mu_s} \qquad ($$

$$+\uparrow \Sigma F_y = 0: \quad N_A + \mu_s N_B - W = 0$$

$$N_A = W - \mu_s N_B$$

$$N_A = W - \frac{1.25\mu_s W}{6 + 2.5\mu_s} \qquad ($$

$$\xrightarrow{+} \Sigma F_x = 0: \quad \mu_s N_A - N_B = 0$$

Substitute for $N_A$ and $N_B$ from Eqs. (1) and (2):

$$\mu_s W - \frac{1.25\mu_s^2 W}{6 + 2.5\mu_s} = \frac{1.25W}{6 + 2.5\mu_s}$$

$$6\mu_s + 2.5\mu_s^2 - 1.25\mu_s^2 = 1.25$$

$$1.25\mu_s^2 + 6\mu_s - 1.25 = 0$$

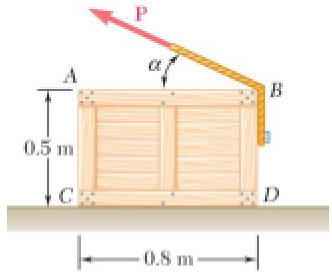
$$\mu_s = 0.2$$

and $\qquad \mu_s = -5 \quad \text{(Discard)}$

$$\mu_s = 0.200$$

**Ep 36.**

A 40-kg packing crate must be moved to the left along the floor without tipping. Knowing that the coefficient of static friction between the crate and the floor is 0.35, draw the free-body diagram needed to determine both the largest allowable value of $\alpha$ and the corresponding magnitude of the force **P**.




**SOLUTION**

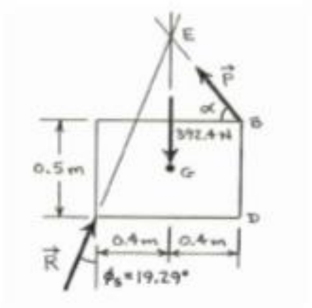
$$W = mg = (40 \text{ kg})(9.81 \text{ m/s}^2) = 392.4 \text{ N}$$

Free-body diagram:

If the crate is about to tip about $C$, contact between crate and ground is at $C$ only, and the reaction **R** is applied at $C$. As the crate is about to slide, **R** must form with the vertical an angle

$$\phi_s = \tan^{-1}\mu_s = \tan^{-1}(0.35) = 19.29^\circ.$$

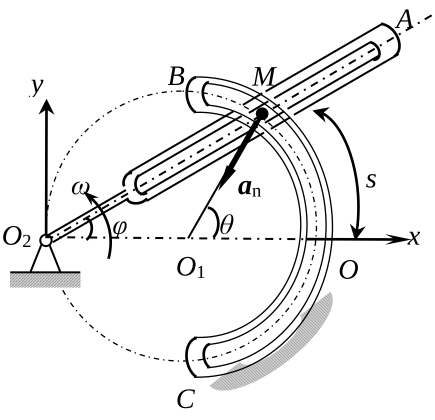
Since the crate is a three-force body, **P** must pass through point $E$ where **R** and **W** intersect. Free body of the crate is then:




Note: After the crate starts moving, $\mu_s$ should be replaced with the lower value $\mu_k$. This will yield a larger value of $\alpha$.

# Kinematics

**Ep 1.** The rocker slide mechanism is shown in Figure 1.The slider $M$ is driven by the rocker $O_2A$ and moves along a fixed circular groove $BC$ with radius $R$. The rotation axis $O_2$ of rocker $O_2A$ is on the circumference of the circular arc groove. If the angle between the joystick and $O_2O$ line according to regular movement with $\varphi=\omega t$, $\omega$ is a constant, try to use cartesian coordinate method and natural method respectively to find the velocity and acceleration of $M$ point.



Ep 1

**SOLUTION:**

1: Since the slider $M$ moves along the known arc path $BC$, it is more convenient to use the natural method to solve it.

Take the starting position of slider $M$ as the origin $O$ of arc coordinates, and specify that its forward and angle φ are in line. According to the geometric relationship in the figure, the motion equation of point $M$ is

$$S=R\theta=2R\varphi=2R\omega t$$

Take the first derivative of the above equation with respect to time to get the velocity of slider $M$:

$$v = \frac{ds}{dt} = 2\omega R$$

The velocity $v$ is a positive constant, indicating that the velocity direction is in the positive direction of the circumferential tangent. The tangential and normal acceleration of the slider $M$ are respectively:

$$a_\tau = \frac{dv}{dt} = 0, \quad a_n = \frac{v^2}{\rho} = 4\omega^2 R$$

The above results indicate that the slider moves in a uniform circular motion with the direction of normal acceleration pointing to $O_1$, as shown in the figure.

2: This problem can also be solved using the cartesian coordinate method.

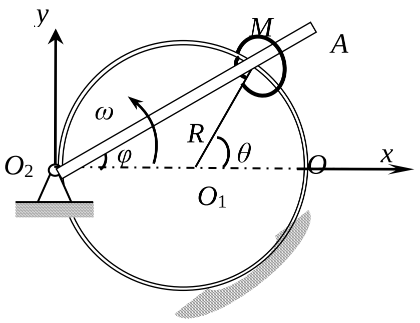
The rectangular coordinate system is established as shown in the figure, and the geometric relationship is as follows:

$$x_M = R + R\cos\theta = R(1+\cos 2\varphi) = R(1+\cos 2\omega t)$$

$$y_M = R\sin\theta = R\sin 2\varphi = R\sin 2\omega t$$

Take the first derivative of the above equation with respect to time to get the velocity of slider M:

$$v_{Mx} = -2\omega R \sin 2\omega t$$
$$v_{My} = 2\omega R \cos 2\omega t$$
$$v_M = \sqrt{v_{Mx}^2 + v_{My}^2} = 2\omega R$$



Ex 1

The velocity is in the positive direction of the circumferential tangent. Also can do it with the direction cosine, now omit it here.

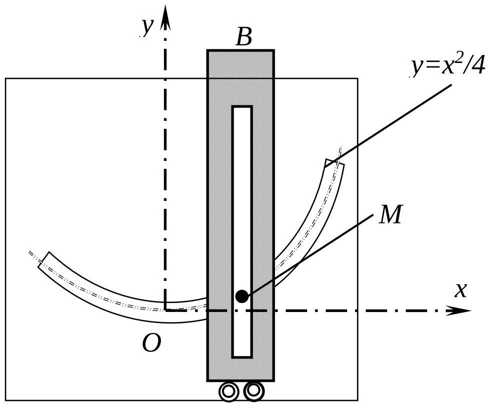
Take the second derivative of the above equation with respect to time, and the acceleration of the slider M is

$$a_{Mx} = -4\omega^2 R \cos 2\omega t$$
$$a_{My} = -4\omega^2 R \sin 2\omega t$$
$$a_M = \sqrt{a_{Mx}^2 + a_{My}^2} = 4\omega^2 R$$

The direction of the acceleration is $O_1$, as shown in the figure.

**Exercise :** As shown in the figure, the small ring M, set in the big ring with radius of *R* and rocker *OA* at the same time, rocker *OA* around *O* axis with an equal angular speed of $2\omega$ rotation. When the motion begins, the joystick is in a horizontal position. Find the M velocity and acceleration of the small ring. ($v$=4ωR,$a_\tau$=0,$a_n$=16$\omega^2$R)

**Ep 2.** For block B with a lead vertical chute, the motion equation of the chute center line is $x$=0.05t$^2$, and drive pin $M$ along a fixed parabolic shape chute, as shown in Figure 2. We know that the equation of the parabola is $y=x^2/4$, where $x$, $y$ are in terms of m. Try to find: (1) when t=5s, the acceleration of pin M; (2) The time when $a_x=a_y$.




Ep 2

**SOLUTION:**

Since pin $M$ is driven by block $B$ and moves in a plane curve, its equation of motion in the $x$ direction is $x=0.05t^2$. By taking the first and second derivatives of the above equation with respect to time, the projection of the velocity and acceleration of pin $M$ on the X-axis are $v_x=0.1t$, $a_x=0.1$m/s$^2$

And because the parabolic equation for a fixed curve slot is $y=x^2/4$, which is the trajectory equation of pin $M$, so take the first derivative of this equation with respect to time, we get $v_y=xv_x/2$, substitute $x$ and $v_x$, we get $v_y=0.0025t^3$, the derivative with respect to time, $a_y=0.0075t^2$, so when $t$=5s,

$$a_y = 0.1875\text{m/s}^2$$

$$a = \sqrt{a_x^2 + a_y^2} = 0.2125\text{m/s}^2$$

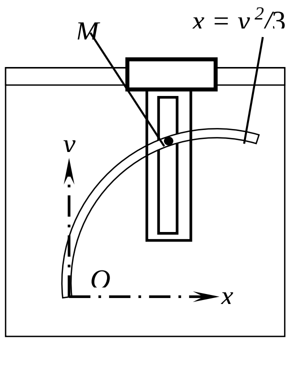
$$\theta = \arctan\frac{a_x}{a_y} = \arctan\frac{0.1}{0.1875} = 28.07^0$$

θ is the positive Angle between a and the Y-axis.

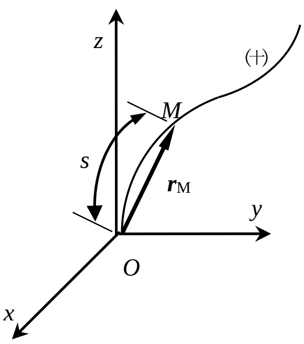
When $a_x=a_y$, we have 0.1=0.0075$t^2$, so $t$=3.65s.

**Exercise :** The lead rod moves to the right at a constant speed of $v_0$ and drives the pin $M$ along the slot of the parabola $x = y^2/3$, as shown in the figure, where $x$ and $y$ are expressed in terms of $m$. Try to find the tangential acceleration $\rho$ of radius and pin $M$ of the trajectory at $y$=2m.（$a_\tau$=0.1688$v_0$m/s$^2$，ρ=6.944m）

**Exercise :** As shown in the figure, the vector diameter of the moving point $M$ is $r_M = 2t\boldsymbol{i} + t^3\boldsymbol{j} + 3t^2\boldsymbol{k}$, where, the unit of length is m, and the unit of time is s. Try to find the tangential acceleration, normal acceleration and radius of curvature of moving point $M$ when t=1s.（$a_\tau$=7.714m/s$^2$，$a_n$=3.535m/s$^2$，ρ=13.86m）

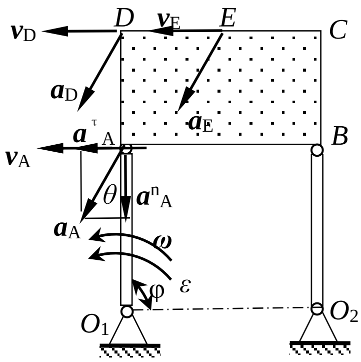



Ex 2




Ex 3

**Ep 3.** As shown in the figure, the system consists of rod $O_1A$, rod $O_2B$ and rectangular plate ABCD. As we know, $O_1A=O_2B=r$, $O_1O_2=AB=CD=2r$, $AD=BC=d$, rod $O_1A$ rotates on axis $O_1$ with an angular acceleration of $\varepsilon=2t\text{rad/s}^2$, where time t is measured in S and length in m. When $t$=0, $\varphi_0=\varphi'_0=0$, calculate the velocity and acceleration of vertex D and middle point E at the upper edge of the rectangular plate when $\varphi=90^0$.



Ep 3

**SOLUTION:**

Find the angular velocity ω of bar $O_1A$

Due to the variable speed motion of bar $O_1A$, When$t$=0, $\varphi_0=\varphi'_0=0$. So

$$\int_0^\omega d\omega = \int_0^t \alpha dt, \qquad \int_0^\varphi d\varphi=\int_0^t \omega dt$$

$$\therefore \quad \omega=\int_0^t 2tdt = t^2, \ \varphi = \int_0^t \omega dt = \frac{t^3}{3}$$

Therefore, when$\varphi=90^0$, the time of experience:

$$t = \sqrt[3]{3\varphi} = \sqrt[3]{3 \cdot \frac{\pi}{2}} = \sqrt[3]{1.5\pi} \ \text{s}$$

The angular velocity of rod $O_1A$ at the instantaneous is:

$$\omega = t^2 = \sqrt[3]{(1.5\pi)^2} \ rad/\text{s}$$

Calculate the velocity and acceleration of point $D$ and $E$ at the middle point of the upper edge

As the rectangular plate moves in curve, it can be written:

$$v_D = v_E = v_A = r\omega = 2.81r \text{ m/s}$$

$$a_A^n = \omega^2 r = r \cdot \sqrt[3]{(1.5\pi)^2} = 7.9r \ \text{m/s}^2$$

$$a_A^\tau = \alpha r = r \cdot 2\sqrt[3]{1.5\pi} = 3.35r \ \text{m/s}^2$$

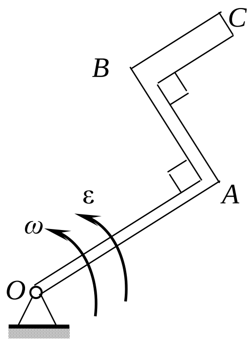
$$a_A = \sqrt{(a_A^n)^2 + (a_A^\tau)^2} = 8.58r \ \text{m/s}^2$$

$$\tan\theta = \frac{a_A^\tau}{a_A^n} = 0.424 \qquad \theta=22.98^0$$

$$a_D = a_E = a_A = 8.58r \ \text{m/s}^2$$

The directions of velocity and acceleration at each point are shown in the figure.

**Exercise :** As shown in the figure, the right Angle folding rod *OABC* rotates within the vertical surface around the *O* axis. As known, *OA*=15cm, *AB*=10cm, *BC*=5cm, the angular acceleration of rotation is $\varepsilon=4t$ rad/s$^2$ (t counts in seconds). If the bar starts to rotate from rest, try to find the velocity and acceleration at points B and C on the bar when t=1s.



Ex 4

$$a_B^n = 20\sqrt{13}\ \text{cm/s}^2$$

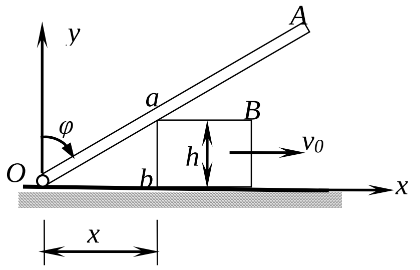
$$a_B^\tau = 20\sqrt{13}\ \text{cm/s}^2$$

$$a_B = \sqrt{(a_B^n)^2 + (a_B^\tau)^2} = 20\sqrt{26}\ \text{cm/s}^2$$

$$v_C = 20\sqrt{5}\ \text{cm/s}$$

$$a_C = 40\sqrt{10}\ \text{cm/s}^2$$

**Ep 4.** Block B moves horizontally and linearly with uniform velocity $v_0$. Bar *OA* can rotate about Axis *O*, and the bar keeps close to the side edge *ab* of the block, as shown in figure 4. Given that the height of the block is *h*, try to find the rotational equation, angular velocity and angular acceleration of bar *OA*.




Ep 4

Solution: Take the coordinates as shown in the figure, φ clockwise from *y* axis to positive, and take $x$=0 as the starting point. At any instant, the coordinate of the contact point between bar *OA* and the lateral edge of the block is *x*. By the question $x=v_0t$, and the triangle relation

$$\tan\varphi = \frac{x}{h} = \frac{v_0t}{h}$$

Therefore, the rotation equation of bar *OA* is

$$\varphi = arctan(\frac{v_0t}{h})$$

The angular velocity of the bar is

$$\omega = \frac{d\varphi}{dt} = \frac{v_0h}{h^2 + (v_0t)^2}$$

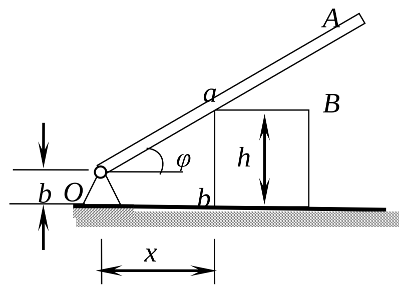
The angular acceleration of the bar is

$$\alpha = \frac{d\omega}{dt} = -\frac{2v_0^3ht}{(h^2 + v_0^2t^2)^2}$$

**Exercise::** In the mechanism shown in the figure, slider B moves to the right as $x$=0.2+0.2$t^2$m, where *t* is measured in seconds. Try to find the angular velocity and angular acceleration of bar *OA* when $x$=0.3m.

$\dot{\varphi} = -0.1375$ rad/s, $\ddot{\varphi} = -6.4788 \times 10^{-2}$ rad/s$^2$

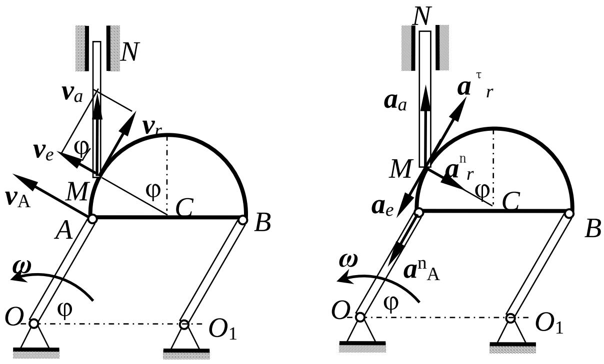
（Turn in counterclockwise）




Ex 5

**Ep 5.** The semicircle plate with radius $R$ is hinged to crank $OA$ and $O_1B$ in the diagram of the planar mechanism.

$OA=O_1B=l, OO_1=AB=2R$。 When crank $OA$ turns, it can drive rod $MN$ to move up and down through the semicircle plate. In the diagram at instantaneous, the angular velocity of crank $OA$ is ω, and the angular acceleration is 0.The included Angle of $OO_1$ and horizontal line and the included Angle of $CM$ and plumb line are $\varphi=60^0$. Try to find the velocity and acceleration of MN at the instantaneous ejector.




Ep 5 (a) Ep 5 (b)

**SOLUTION:**

The relative motion (trajectory) is the curve along the edge of the semicircle plate; implicated motion is curve translation.

1） Calculate the speed of $MN$ ejecting rod

According to the velocity composition theorem

$$\boldsymbol{v}_a = \boldsymbol{v}_e + \boldsymbol{v}_r$$

Because the semicircle moves, so $v_e=v_A=\omega l$, the velocity synthesis diagram (velocity parallelogram) is shown in figure 5 (a).

The figure shows

$$v_a = \frac{v_e}{cos\varphi} = \frac{\omega l}{cos60^0} = 2\omega l$$

$$v_r = v_e tan\varphi = \omega l tan60^0 = \sqrt{3}\omega l$$

Therefore, the velocity of $MN$ bar is $v_{MN}=v_a=2\omega l$, and the direction of the lead is straight up.

2） Find the acceleration of $MN$ rod

According to the synthesis theorem of acceleration involving motion is translational

$$\boldsymbol{a}_a = \boldsymbol{a}_e + \boldsymbol{a}_r^n + \boldsymbol{a}_r^{\tau}$$

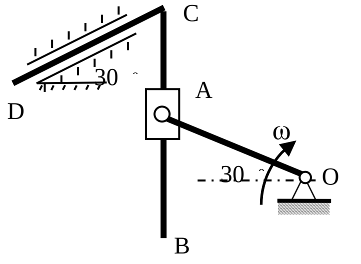
The acceleration vector diagram is shown in Figure 5(b). Among them

$$a_e = a_A = \omega^2 l, a_r^n = \frac{v_r^2}{R}$$

Projection of acceleration vector equation along $CM$ , it can be obtained

$$a_a \cos\varphi = a_r^n = \frac{v_r^2}{R},$$

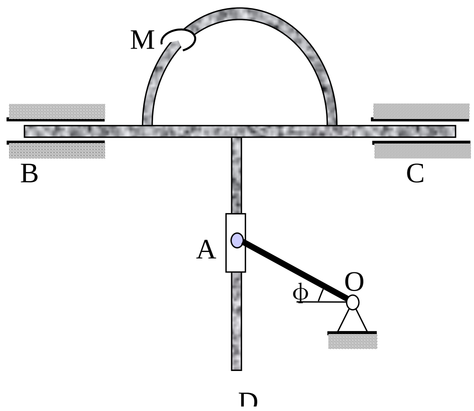
$$\therefore\ a_{MN} = a_a = \frac{v_r^2}{R\cos 60^0} = \frac{6\omega^2 l^2}{R}$$, it's going to plumb down.




Ex 6

**Exercise:** In the diagram, the crank is known to rotate about *O* axis at an even angular velocity ω, *OA*=R. At the point in the figure, *CB* section is vertical and *DCB* is a whole. *DC* section slides in the slide with an inclination of $30^0$. Calculate the velocity and acceleration of this instantaneous broken rod *DCB*.

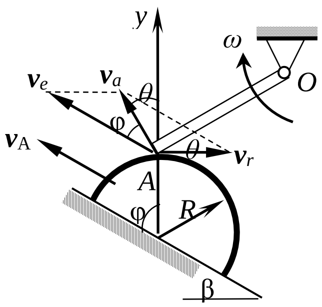
( $v_e = \frac{\sqrt{3}\omega R}{3}$, $a_e = a_a^n = \omega^2 R$ )



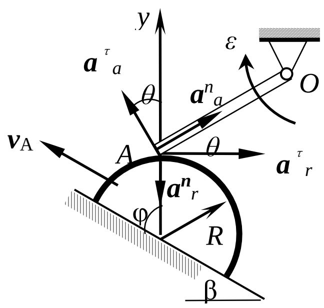

Ex 7

**Exercise:** The Graphic flat mechanism. Crank *OA* rotates around the axis with the regularity of $\varphi = \frac{\pi t}{18}(rad)$, which drives the *T*-bar *BCD* to move around. The semi-circular plate with radius *R* is firmly connected to the *T*-bar. The small ring *M* moves along the semicircle plate according to the law of $O_1M = s = \pi t^2$, *OA*=*R*=18cm, try to find: when t=3s (that is, the small ring *M* to the highest point of the semicircle plate), the speed and acceleration of the small ring *M*.

**Ep 6.** The semicircular CAM moves along an inclined plane with an inclination of $\beta=30^0$, driving $OA$ to swing around $O$ axis. As known, $R$=10cm, $OA$=20cm, at the position shown in Figure 6, the angle between $OA$ and horizontal line is $\theta=30^0$, $\varphi=60^0$, the CAM speed $v_A$=60cm/s, and the acceleration is 0.Try to find the angular velocity and angular acceleration of the instantaneous $OA$ bar.




Ep 6 (a)




Ep 6 (b)

**SOLUTION:**

Take point $A$ on the $OA$ bar as the moving point, the moving system is firmly connected to the CAM, the static system is firmly connected to the ground, and the absolute motion (track) is the circumference;The relative motion (trajectory) is the curve along the edge of the semicircle;The motion involved is translation.

1）Velocity analysis

According to the velocity composition theorem

$$\boldsymbol{v}_a = \boldsymbol{v}_e + \boldsymbol{v}_r$$

Because the semicircular CAM moves in translation, so $v_e=v_A$=60cm/s, The resultant velocity diagram (parallelogram of velocity) is shown in figure 6 (a), and the resultant velocity vector is projected onto the Y-axis:

$$v_a cos30^0 = v_e cos60^0$$

$$v_a = v_e tan30^0 = 34.64\text{cm/s}$$

$$\omega_0 = \frac{v_a}{OA} = 1.73\text{rad/s}$$

By geometric relations

$$v_r = v_e = 34.64\text{cm/s}$$

2）Acceleration analysis

According to the synthesis theorem of acceleration involving motion is translational

$$\boldsymbol{a}_a = \boldsymbol{a}_e + \boldsymbol{a}_r^n + \boldsymbol{a}_r^\tau$$

The acceleration vector diagram is shown in figure 6(b). Among them:

$$a_e = 0,\ a_a^n = \frac{v_a^2}{OA} = 60\text{cm/s}^2,\ a_r^n = \frac{v_r^2}{R} = 120\text{cm/s}^2$$

The acceleration vector equation is projected onto the *y*-axis, and

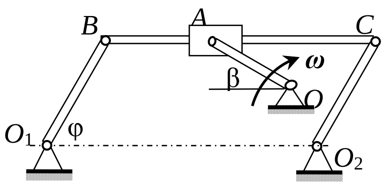
$$a_a^\tau \cos 30^0 + a_a^n \sin 30^0 = -a_r^n,$$

$$\therefore\ a_a^\tau = -173.2\text{cm/s}^2$$

$$\alpha = -8.66\text{rad/s}^2$$

Turn clockwise.

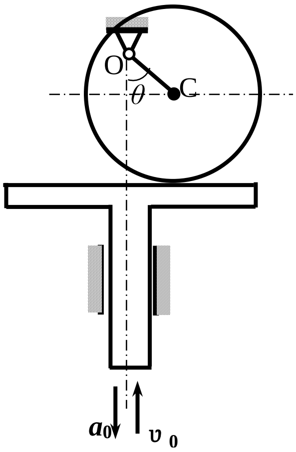
**Exercise:** Screen mechanism as shown in the figure, the crank $OA=l_1=15$cm, the sleeve *A* is hinged to the *OA* bar, $O_1B = O_2C = 20\sqrt{3}$cm, and $O_1B // O_2C$. At the point in the figure $\varphi=60^0$, $\beta=30^0$, $\omega=2$ rad/s，$\alpha=0$。Try to find the angular velocity and angular acceleration of this instantaneous $O_1B$ bar.



Ex 8

（$\omega_1=1.5$ rad/s, counterclockwise; $\alpha_1=-2.16$ rad/s$^2$, counterclockwise.）

**Exercise:** The eccentricity of the eccentric wheel is $OC=e$. When the Angle between *OC* and the plumb line is $\theta$, the velocity and acceleration of the T-shaped push rod are $v_\circ$ and $a_\circ$, and the direction is shown in the figure. Find the angular velocity and angular acceleration of the instantaneous eccentric wheel.



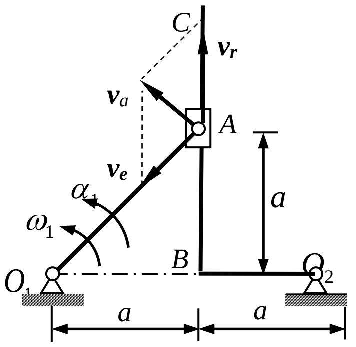
Ex 9

$$\omega = \frac{v_0}{e\sin\theta}$$

（$\alpha = \dfrac{a_0}{e\sin\theta} + \dfrac{v_0^2\cos\theta}{e^2\sin^3\theta}$ , the angular velocity rotates counterclockwise and the angular acceleration rotates clockwise）

**Ep 7.** As shown in figure 7, the angular velocity and angular acceleration of crank $O_1A$ are known as $\omega_1$ and $\alpha_1$ respectively, and $a$ is known. Find the angular velocity and angular acceleration $\omega_2$ and $\alpha_2$ of the right Angle bar $O_2BC$ at the position shown in the diagram.

**SOLUTION:**



Ep 7(a)

Take A as the moving point, the moving system is firmly connected to $O_2BC$, the static system is firmly connected to the ground, and the absolute motion (trajectory) is the circumference. The relative motion (trajectory) is plumb vertical line; the motion involved is a fixed axis rotation.

1）Velocity analysis is shown in Figure 7 (a)

According to the velocity composition theorem

$$\boldsymbol{v}_a = \boldsymbol{v}_e + \boldsymbol{v}_r$$

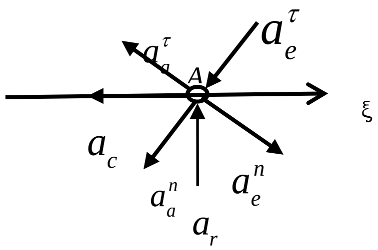
In which $v_a = \sqrt{2}\omega_1 a$

From the geometric relationship

$$v_e = v_a = \sqrt{2}\omega_1 a \quad v_r = \sqrt{2}v_a = 2\omega_1 a$$

$$\omega_{O_2BC} = \frac{v_e}{\sqrt{2}a} = \omega_1$$ counterclockwise.

2）Acceleration analysis, as shown in Figure 7 (b)



Ep 7(b)

Calculate the Coriolis acceleration $a_C = 2\omega_e v_r = 4\omega_1^2 a$

According to the acceleration synthesis theorem:

$$a_a^n + a_a^\tau = a_e^n + a_e^\tau + a_r + a_C$$

The equation is obtained by projection on the ξ-axis

$$-a_a^n \cos 45° - a_a^\tau \cos 45° = a_e^n \cos 45° - a_e^\tau \cos 45° - a_C$$

In which $a_e^\tau = \alpha_2 \sqrt{2}a$

Solve it $\alpha_2 = 2\omega_1^2 - \alpha_1$

Counterclockwise.

**Ep 8.** Known the CAM in the CAM rod mechanism is an eccentric round wheel, as shown in Figure 8, its radius is R, eccentricity is E, and in speed of $\omega$ for equal angular speed rotation. When $\angle OCA = 90^\circ$, calculate the speed and acceleration of rod AB.

**SOLUTION:**

Point *A* on the jacking rod *AB* is the moving point, the dynamic system is fixed with the eccentric disk, the static system is fixed with the ground, and the absolute motion (trajectory) is a plumb vertical line. The relative motion (trajectory) is the circle; The implicated motion is the fixed axis rotation of the dynamic system around *O* axis.

1）Velocity analysis is shown in figure. 8 (a)

According to the velocity composition theorem

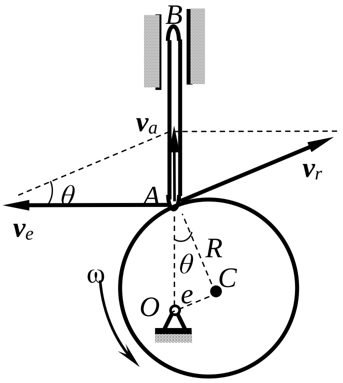
$$\boldsymbol{v}_a = \boldsymbol{v}_e + \boldsymbol{v}_r$$

In which $v_e = \sqrt{R^2 + e^2}\omega$

From the geometric relationship, we get

$$v_{AB} = v_a = v_e \tan\theta = \frac{e}{R}\sqrt{R^2 + e^2}\omega$$

$$v_r = \frac{v_e}{\cos\theta} = \frac{R^2 + e^2}{R}\omega$$



Ep 8 (a)

2）The acceleration analysis, as shown in Figure 8 (b), calculates the known acceleration

$$a_C = 2\omega_e v_r = 2\frac{R^2 + e^2}{R}\omega^2$$

$$a_r^n = \frac{v_r^2}{R} = \frac{(R^2 + e^2)^2}{R^3}\omega^2$$

$$a_e = \sqrt{R^2 + e^2}\omega^2$$

According to the acceleration synthesis theorem:

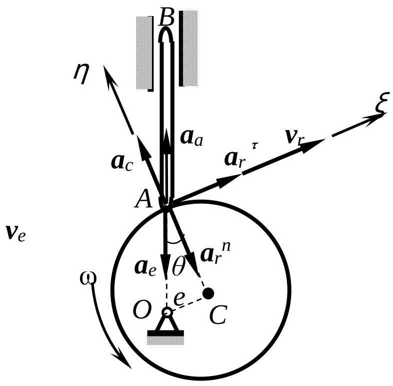
$$\boldsymbol{a}_a = \boldsymbol{a}_r^n + \boldsymbol{a}_r^\tau + \boldsymbol{a}_e + \boldsymbol{a}_C$$

Projecting the equation on the η-axis:

$$a_a = \frac{v_r^2}{R} = -\frac{e^4\sqrt{R^2 + e^2}}{R^4}\omega^2$$

Projecting the equation on the ξ-axis:

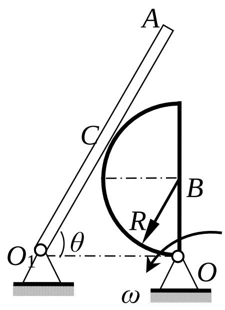
$$a_r^\tau = \frac{v_r^2}{R} = (1 - \frac{e^4}{R^4})e\omega^2$$



Ep 8 (b)

There are other ways to solve this problem. Let the reader practice by himself.

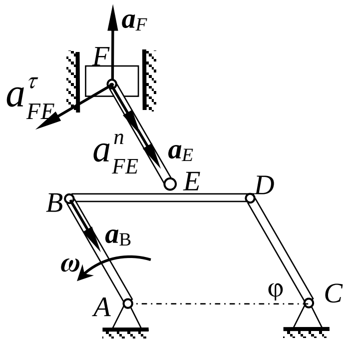
**Exercise:** In the mechanism shown in the figure, the rocker $O_1A$ is always in contact with the semicircular CAM $B$ with a radius of R. When the CAM $B$ swings back and forth around the axis $O$, the rocker will swing around the axis $O_1$. Suppose $OB \perp OO_1$, $\theta=60^0$, CAM $B$ angular velocity is $\omega$, angular acceleration is 0. Try to find the angular velocity and angular acceleration of the instantaneous rocker $O_1B$.




Ex 10

( $\omega_1 = \dfrac{\omega}{2}$ , $\alpha_1 = \dfrac{\sqrt{3}}{12}\omega^2$ )

**Ep 9.** The mechanism is shown in Figure 9, $AB = CD = EF = l$, and $AB// CD//EF$. At the position φ=$60^0$, rod $AB$ rotates at a uniform angular speed ω. Try to find the angular velocity, angular acceleration of the instantaneous $EF$ bar and the acceleration of the slider $F$.



Ep 9

**SOLUTION:**

The bar $EF$ does plane motion. It can be seen from the velocity orientation of two points $E$ and $F$, and point $F$ is its instantaneous center.
Because the bar $BD$ does translation, so

$$v_E = v_B = l\omega$$

$$\therefore \ \omega_{EF} = \frac{v_E}{EF} = \frac{l\omega}{l} = \omega$$

Select basic point E and analyze point F. The acceleration vector diagram is shown in Figure. 9. According to the acceleration synthesis theorem:

$$\boldsymbol{a}_F = \boldsymbol{a}^n_{FE} + \boldsymbol{a}^{\tau}_{FE} + \boldsymbol{a}_E$$

In which:

$$a_E = a_B = l\omega^2$$

$$a^n_{FE} = \omega^2_{FE} l = l\omega^2$$

Projection of acceleration vector equation in $EF$ direction, it can be obtained as follows:

$$a_F \cos 30^0 = -a_E - a^n_{FE}$$

$$\therefore \ a_F = -\frac{a_E + a^n_{FE}}{\cos 30^0} = \frac{\omega^2 l + \omega^2 l}{\cos 30^0} = \frac{4\sqrt{3}}{3}\omega^2 l$$

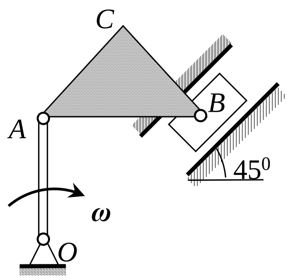
Projection of acceleration vector equation in $a^{\tau}_{FE}$ direction, it can be obtained as follows:

$$- a_F \cos 60^0 = a^{\tau}_{FE}$$

$$\therefore \ a^{\tau}_{FE} = -\frac{2\sqrt{3}}{3}\omega^2 l$$

$$\therefore \ \alpha_{FE} = \frac{a^{\tau}_{FE}}{FE} = \frac{2\sqrt{3}}{3}\omega^2$$ ( Turn clockwise.)

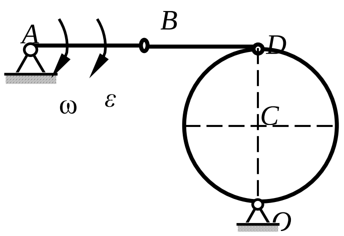
**Exercise:** As shown in the figure, crank *OA* rotates around *O* axis with uniform angular speed $\omega$. Known: $OA=r$, $AB=4r$. Try to find the angular velocity, angular acceleration of triangle *ABC* and the acceleration of slider *B* when crank *OA* is perpendicular to the bottom edge of triangle AB.



Ex 11

( $\omega_{\square} = \dfrac{\omega}{4}$ , $\therefore\ a_B = \dfrac{\sqrt{2}}{4}\omega^2 r$ $\therefore\ \alpha_{\square} = \dfrac{5}{16}\omega^2$ )

**Ep 10.** Rod *AB*, *BD* and disk *C* are hinged together and supported as shown in figure. 10 (a). Known: $AB=BD=OD=2R$, it is shown that instantaneous *A*, *B* and *D* are in the same horizontal and linear position, while *OD* lead position. At this moment, the angular velocity of *AB* bar is $\omega_o$ and the angular acceleration is $\alpha_o$. Calculate the angular velocity and angular acceleration of the disk around axis *O* at this moment.



Ep 10(a)

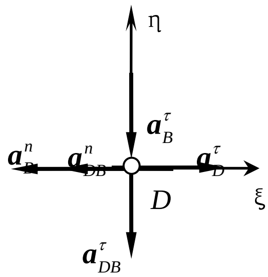
**SOLUTION:**

The bar *BD* moves in a plane. According to the velocity azimuth of two points *B* and *D*, point *D* is their instantaneous center. So

$$v_D = 0 \quad v_B = 2\omega_0 R$$

$$\omega_{OD} = \frac{v_D}{OD} = 0$$

Select base point B and analyze point D, whose acceleration vector diagram is shown in Figure. 10 (b). According to the acceleration synthesis theorem:



Ep 10(b)

$$\boldsymbol{a}_D = \boldsymbol{a}_{DB}^n + \boldsymbol{a}_{DB}^\tau + \boldsymbol{a}_B$$

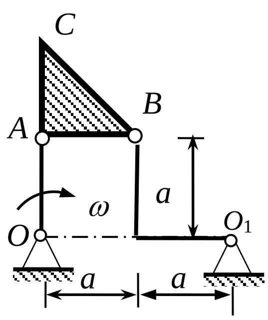
In which：

$$a_B^n = 2\omega_0^2 R \qquad a_{DB}^n = 2\omega_0^2 R$$

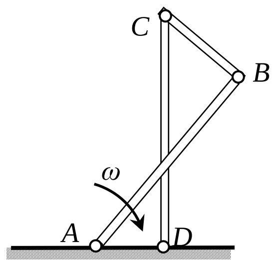
Projection of acceleration vector equation in ξ-axis direction, it can be obtained as follows:

$$a_D^\tau = 4\omega_0^2 R$$

$$\alpha_{OD} = \frac{a_D^\tau}{OD} = 2\omega_0^2$$



Ex 12



Ex 13

**Exercise:** As shown in the figure, crank *OA* rotates about *O* axis at a uniform angular velocity $\omega$. Find the angular velocity and angular acceleration of the bending rod $O_1B$.

( $\omega_{01} = \omega$, $\alpha_{01} = 2\omega^2$)

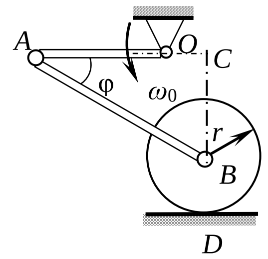
Ex **13:** The parallelogram mechanism is shown in the figure. Known: $AB=CD=l=40$cm, $BC=AD=b=20$cm. The crank *AB* rotates about the axis *A* with an uniform angular velocity $\omega$=3rad/s. Try to find the angular velocity and angular acceleration of bar *BC* when $CD \perp AD$.

（ $\omega_{BC}$ = 8rad/s, $\alpha_{BC}$ = 20rad/s$^2$ ）

**Ep 11.** In the mechanism is shown in Figure. 11, the crank *OA* length is *R*, the bar *AB* length is *l*, and the radius of the pure rolling round wheel is *r* at the uniform angular velocity $\omega$ around *O*-axis. Try to find the angular velocity and angular acceleration of bar *AB* and round wheel when *OA* is at horizontal, and $\varphi$=30$^0$.

**SOLUTION:**

The instantaneous center of bar *AB* and circular rotation motion is *C* and *D* respectively.



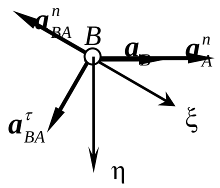
Ep 11(a)

$$\because v_A = \omega_0 R, \quad \therefore \omega_{AB} = \frac{v_A}{AC} = \frac{v_A}{l\cos\varphi} = \frac{2\sqrt{3}\omega_0 R}{3l}$$

$$\because v_B = \omega_{AB} \cdot CB = \omega_{AB} \cdot AB \cdot \sin\varphi = \frac{\sqrt{3}\omega_0 R}{3}$$

$$\therefore \omega_B = \frac{v_B}{r} = \frac{\sqrt{3}\omega_0 R}{3r}$$

Select base point *A* and analyze point *B*, whose acceleration vector diagram is shown in Figure 11(b).



Ep 11(b)

In which, $a_A^n = \omega_0^2 R, \ a_{BA}^n = \omega_{AB}^2 \cdot AB = \dfrac{4\omega_0^2 R^2}{3l}$

According to the acceleration synthesis theorem:

$$\boldsymbol{a}_B = \boldsymbol{a}_{BA}^n + \boldsymbol{a}_{BA}^\tau + \boldsymbol{a}_A^n$$

Project the above equation in ξ-axis direction, it can be obtained as follows:

$$a_B \cdot \cos 30^0 = a_A^n \cdot \cos 30^0 - a_{BA}^n$$

$$\therefore a_B = \frac{a_A^n \cdot \cos 30^0 - a_{BA}^n}{\cos 30^0} = \omega_0^2 R(1 - \frac{8\sqrt{3}R}{9l})$$

Project the above equation in η-axis direction, it can be obtained as follows:

$$0 = a_{BA}^\tau \cdot \cos 30^0 - a_{BA}^n \cdot \cos 30^0$$

$$\therefore a_{BA}^\tau = a_{BA}^n = \frac{4}{3l}\omega_0^2 R^2$$

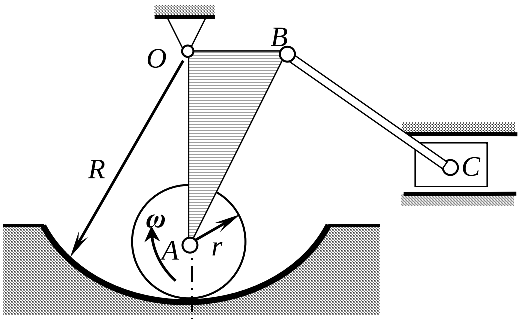
$$\therefore \alpha_{BA} = \frac{a_{BA}^\tau}{BA} = \frac{4}{3l^2}\omega_0^2 R^2$$

$$\alpha_B = \frac{a_B}{r} = \omega_0^2 \frac{R}{r}(1 - \frac{8\sqrt{3}R}{9l})$$

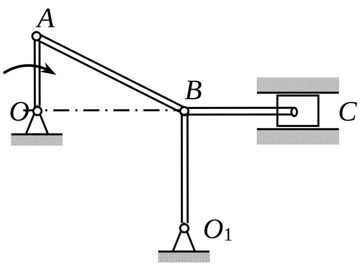
**Exercise:** As shown in the figure, the right triangle plate *OABC* rotates around the *O* axis. Point *A* is hinged to A disk with radius *r*, and the disk rolls purely in a surface with radius *R*. Known: *OA*=*BC*=30cm, *r*=10cm, *R*=40cm. At the moment in the figure, the angular velocity of the disk is $\omega$=2rad/s, *OA* plumb, *AB*⊥*BC*. Try to find the velocity and acceleration of slider *C*.圆盘角速度，*OA* 铅垂，。试求滑块 *C* 的速度和加速度。

($v_C$=5cm/s, $a_C$=11.6cm/s$^2$)

**Exercise:** The mechanism is shown in the figure. It is known that the length of *OA* is *r*, the angular velocity is $\omega$, the length of $O_1B$ is 5*r*, the length of *AB* and *BC* is 6*r*. When *OA* and $O_1B$ are upright, try to find the velocity and acceleration at point *C*.



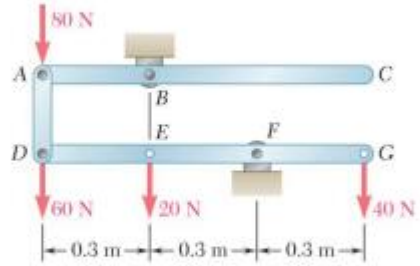

Ex 14



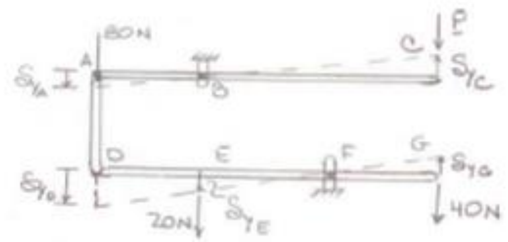

Ex 15

**Ep 12.**

Determine the vertical force **P** that must be applied at $C$ to maintain the equilibrium of the linkage.




**SOLUTION**



Assume $\delta y_A \downarrow$

$$\delta y_C = 2\delta y_A \uparrow$$

$$\delta y_D = \delta y_A \downarrow$$

$$\delta y_E = \frac{1}{2}\delta y_D = \frac{1}{2}\delta y_A \downarrow$$

$$\delta y_G = \delta y_E = \frac{1}{2}\delta y_A \uparrow$$

Also $\quad \delta\theta = \dfrac{\delta y_A}{a} \quad\quad a = 0.3 \text{ m}$

<u>Virtual Work</u>: We apply both $P$ and $M$ to member $ABC$.

$$\delta U = 80\delta y_A - P\delta y_C - M\delta\theta + 60\delta y_D + 20\delta y_E - 40\delta y_G = 0$$

$$80\delta y_A - P(2\delta y_A) - M\left(\frac{\delta y_A}{a}\right) + 60\delta y_A + 20\left(\frac{1}{2}\delta y_A\right) - 40\left(\frac{1}{2}\delta y_A\right) = 0$$

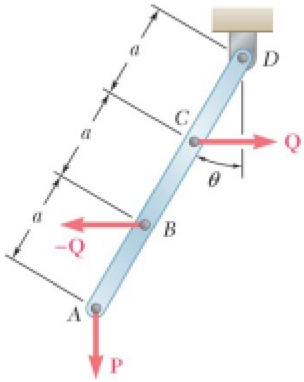
$$80 - 2P - \frac{M}{a} + 60 + 10 - 20 = 0$$

$$2P + \frac{M}{a} = 130 \text{ N} \qquad (1)$$

Now from Eq. (1) for $M = 0$ $\quad 2P = 130 \text{ N} \quad\quad \mathbf{P} = 65.0 \text{ N}\downarrow$

**Ep 13.**

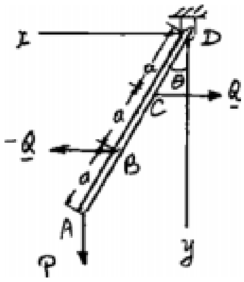
Rod $AD$ is acted upon by a vertical force **P** at end $A$, and by two equal and opposite horizontal forces of magnitude $Q$ at points $B$ and $C$. Derive an expression for the magnitude $Q$ of the horizontal forces required for equilibrium.




**SOLUTION**

We have

$$x_C = a\sin\theta$$
$$\delta x_C = a\cos\theta\delta\theta$$
$$x_B = 2a\sin\theta$$
$$\delta x_B = 2a\cos\theta\delta\theta$$
$$y_A = 3a\cos\theta$$
$$\delta y_A = -3a\sin\theta\delta\theta$$




Virtual Work: We note that **P** tends to increase $y_A$ and −**Q** tends to increase $x_B$, while **Q** tends to decrease $x_C$. Therefore

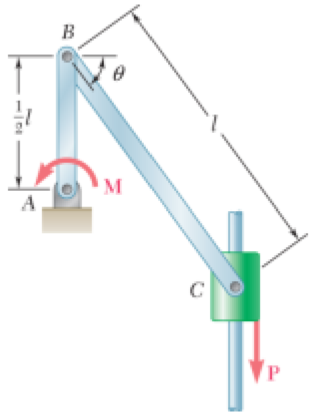
$$\delta U = P\delta y_A + Q\delta x_B - Q\delta x_C = 0$$
$$= P(-3a\sin\theta\delta\theta) + Q(2a\cos\theta\delta\theta) - Q(a\cos\theta\delta\theta) = 0$$

$$Q\cos\theta = 3P\sin\theta'$$

$$Q = 3P\tan\theta$$

**Ep 14.**

Knowing that the coefficient of static friction between collar $C$ and the vertical rod is 0.40, determine the magnitude of the largest and smallest couple **M** for which equilibrium is maintained in the position shown, when $\theta = 35°$, $l = 600$ mm, and $P = 300$ N.




**SOLUTION**

From the analysis of Problem 10.50, we have

$$M_{max} = \frac{Pl}{2(\tan\theta + \mu_s)}$$

With $\theta = 35°, \quad l = 0.6\text{ m}, \quad P = 300\text{ N}$

$$M_{max} = \frac{(300\text{ N})(0.6\text{ m})}{2(\tan 35° - 0.4)}$$

$$= 299.80\text{ N}\cdot\text{m} \qquad M_{max} = 300\text{ N}\cdot\text{m}$$

For $M_{min}$, motion of $C$ impends downward and $F$ acts upward. The equations of Problem 10.50 can still be used if we replace $\mu_s$ by $-\mu_s$. Then

$$M_{min} = \frac{Pl}{2(\tan\theta + \mu_s)}$$

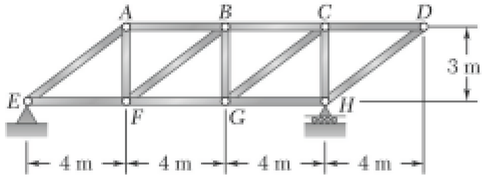
Substituting,

$$M_{min} = \frac{(300\text{ N})(0.6\text{ m})}{2(\tan 35° + 0.4)}$$

$$= 81.803\text{ N}\cdot\text{m} \qquad M_{min} = 81.8\text{ N}\cdot\text{m}$$

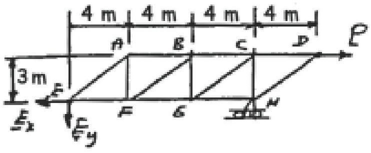
## Ep 15.

Determine the horizontal movement of joint $D$ if the length of member $BF$ is increased by 38 mm. (See the hint for Problem 10.57.)



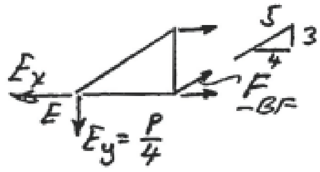

### SOLUTION

Apply horizontal load $P$ at $D$.



$$+\circlearrowleft \Sigma M_H = 0: \quad P(3\text{ m}) - E_y(12\text{ m}) = 0$$

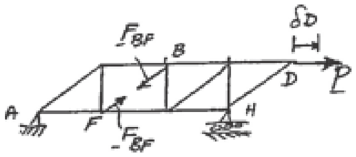
$$\mathbf{E}_y = \frac{P}{4}\downarrow$$



$$+\uparrow \Sigma F_y = 0: \quad \frac{3}{5}F_{BF} - \frac{P}{4} = 0$$

$$F_{BF} = \frac{5}{12}P$$

We remove member $BF$ and replace it with forces $\mathbf{F}_{BF}$ and $-\mathbf{F}_{BF}$ at pins $F$ and $B$, respectively. Denoting the virtual displacements of points $B$ and $F$ as $\delta\mathbf{r}_B$ and $\delta\mathbf{r}_F$, respectively, and noting that $\mathbf{P}$ and $\overline{\delta D}$ have the same direction, we have

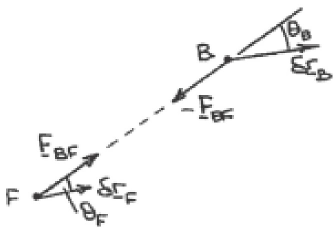


Virtual Work:

$$\delta U = 0: \qquad P\delta D + \mathbf{F}_{BF}\cdot\delta\mathbf{r}_F + (-\mathbf{F}_{BF})\cdot\delta\mathbf{r}_B = 0$$

$$P\delta D + F_{BF}\delta r_F\cos\theta_F - F_{BF}\delta r_B\cos\theta_B = 0$$

$$P\delta D - F_{BF}(\delta r_B\cos\theta_B - \delta r_F\cos\theta_F) = 0$$



where $(\delta r_B\cos\theta_B - \delta r_F\cos\theta_F) = \delta_{BF}$, which is the change in length of member $BF$. Thus,

$$P\delta D - F_{BF}\delta_{BF} = 0$$

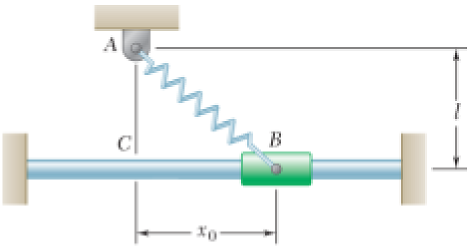
$$P\delta D - \left(\frac{5}{12}P\right)(38\text{ mm}) = 0$$

$$\delta D = 15.8\text{ mm}$$

$$\delta D = 15.8\text{ mm}$$

**Ep 16.**

A spring $AB$ is attached to a support at $A$ and to a collar. The unstretched length of the spring is $l$. Knowing that the collar is released from rest at $x = x_0$ and has an acceleration defined by the relation $a = -100(x - lx/\sqrt{l^2 + x^2})$, determine the velocity of the collar as it passes through Point $C$.




**SOLUTION**

Since $a$ is function of $x$,

$$a = v\frac{dv}{dx} = -100\left(x - \frac{lx}{\sqrt{l^2 + x^2}}\right)$$

Separate variables and integrate:

$$\int_{v_0}^{v_f} v\,dv = -100\int_{x_0}^{0}\left(x - \frac{lx}{\sqrt{l^2 + x^2}}\right)dx$$

$$\frac{1}{2}v_f^2 - \frac{1}{2}v_0^2 = -100\left(\frac{x^2}{2} - l\sqrt{l^2 + x^2}\right)\Bigg|_{x_0}^{0}$$

$$\frac{1}{2}v_f^2 - 0 = -100\left(-\frac{x_0^2}{2} - l^2 + l\sqrt{l^2 + x_0^2}\right)$$

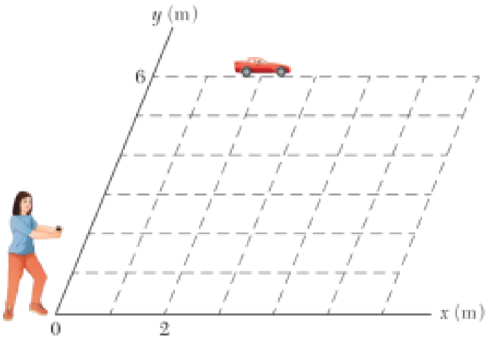
$$\frac{1}{2}v_f^2 = \frac{100}{2}(-l^2 + x_0^2 - l^2 - 2l\sqrt{l^2 + x_0^2})$$

$$= \frac{100}{2}(\sqrt{l^2 + x_0^2} - l)^2$$

$$v_f = 10(\sqrt{l^2 + x_0^2} - l)$$

**Ep 17.**

A girl operates a radio-controlled model car in a vacant parking lot. The girl's position is at the origin of the $xy$ coordinate axes, and the surface of the parking lot lies in the $x$-$y$ plane. The motion of the car is defined by the position vector $\vec{r} = (2+2t^2)\hat{i} + (6+t^3)\hat{j}$ where $r$ and $t$ are expressed in meters and seconds, respectively. Determine (a) the distance between the car and the girl when $t = 2$ s, ($b$) the distance the car traveled in the interval from $t = 0$ to $t = 2$ s, ($c$) the speed and direction of the car's velocity at $t = 2$ s, (d) the magnitude of the car's acceleration at $t = 2$ s.




## SOLUTION

Given: $\mathbf{r} = (2+2t^2)\mathbf{i} + (6+t^3)\mathbf{j}$

*(a)* At t=2s $\quad \mathbf{r}(2) = 10\mathbf{i} + 14\mathbf{j}$ m

$$|\mathbf{r}(2)| = \sqrt{10^2 + 14^2} \text{ m} \qquad |\mathbf{r}(2)| = 17.20 \text{ m}$$

*(b)* The car is traveling on a curved path. The distance that the car travels during any infinitesimal interv

is given by: $\quad ds = \sqrt{dx^2 + dy^2}$

where: $\quad dx = 4t\,dt$ and $dy = 3t^2 dt$

Substituting $\quad ds = \sqrt{16t^2 + 9t^4}\,dt$

Integrating: $\quad \int_0^s ds = \int_0^2 t\sqrt{16 + 9t^2}\,dt$

$$s = \frac{1}{27}\left(16 + 9t^2\right)^{3/2}\Big|_0^2 \qquad s = 11.52 \text{ m}$$

*(c)* Velocity can be found by $\quad \mathbf{v} = \dfrac{d\mathbf{r}}{dt}$

$$\mathbf{v} = 4t\mathbf{i} + 3t^2\mathbf{j} \text{ m/s}$$

At t=2 s $\quad \mathbf{v} = 8\mathbf{i} + 12\mathbf{j}$ m/s $\qquad |\mathbf{v}| = 14.42 \text{ m/s} \measuredangle 56.31°$

*(d)* Acceleration can be found by $\mathbf{a} = \dfrac{d\mathbf{v}}{dt}$

$$\mathbf{a} = 4\mathbf{i} + 6t\mathbf{j} \text{ m/s}^2$$

At $t$=2 s $\quad \mathbf{a} = 4\mathbf{i} + 12\mathbf{j}$ m/s$^2$ $\qquad |\mathbf{a}| = 12.65 \text{ m/s}^2$

**Ep 18.**

A monorail train starts from rest on a curve of radius 400 m and accelerates at the constant rate $a_t$. If the maximum total acceleration of the train must not exceed $1.5\ \text{m/s}^2$, determine (*a*) the shortest distance in which the train can reach a speed of 72 km/h, (*b*) the corresponding constant rate of acceleration $a_t$.

## SOLUTION

When $v = 72\ \text{km/h} = 20\ \text{m/s}$ and $\rho = 400$ m,

$$a_n = \frac{v^2}{\rho} = \frac{(20)^2}{400} = 1.000\ \text{m/s}^2$$

But

$$a = \sqrt{a_n^2 + a_t^2}$$

$$a_t = \sqrt{a^2 - a_n^2} = \sqrt{(1.5)^2 - (1.000)^2} = \pm 1.11803\ \text{m/s}^2$$

Since the train is accelerating, reject the negative value.

(*a*) Distance to reach the speed.

Let

$$v_0 = 0$$

$$x_0 = 0$$

$$v_1^2 = v_0^2 + 2a_t(x_1 - x_0) = 2a_t x_1$$

$$x_1 = \frac{v_1^2}{2a_t} = \frac{(20)^2}{(2)(1.11803)}$$

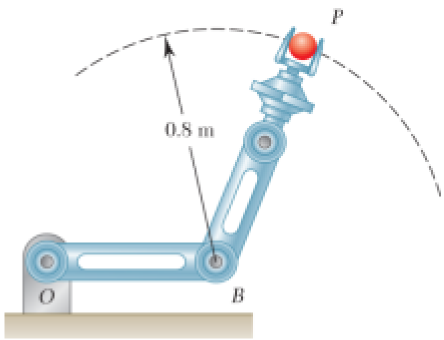
$$x_1 = 178.9\ \text{m}$$

(*b*) Corresponding tangential acceleration.

$$a_t = 1.118\ \text{m/s}^2$$

**Ep 19.**

A robot arm moves so that $P$ travels in a circle about Point $B$, which is not moving. Knowing that $P$ starts from rest, and its speed increases at a constant rate of 10 $\text{mm/s}^2$, determine ($a$) the magnitude of the acceleration when $t$= 4 s, ($b$) the time for the magnitude of the acceleration to be 80 $\text{mm/s}^2$.




## SOLUTION

Tangential acceleration: $a_t = 10 \text{ mm/s}^2$

Speed: $v = a_t t$

Normal acceleration: $a_n = \dfrac{v^2}{\rho} = \dfrac{a_t^2 t^2}{\rho}$

where $\rho = 0.8 \text{ m} = 800 \text{ mm}$

($a$) When $t = 4$ s $\quad v = (10)(4) = 40 \text{ mm/s}$

$$a_n = \frac{(40)^2}{800} = 2 \text{ mm/s}^2$$

Acceleration: $a = \sqrt{a_t^2 + a_n^2} = \sqrt{(10)^2 + (2)^2}$

$$a = 10.20 \text{ mm/s}^2$$

($b$) Time when $a = 80 \text{ mm/s}^2$

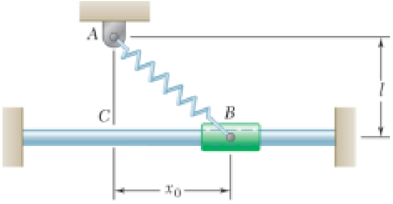
$$a^2 = a_n^2 + a_t^2$$

$$(80)^2 = \left[\frac{(10)^2 t^2}{800}\right]^2 + 10^2 \qquad t^4 = 403200 \text{ s}^4$$

$$t = 25.2 \text{ s}$$

## Ep 20.

A spring $AB$ of constant $k$ is attached to a support at $A$ and to a collar of mass $m$. The unstretched length of the spring is $\ell$. Knowing that the collar is released from rest at $x = x_0$ and neglecting friction between the collar and the horizontal rod, determine the magnitude of the velocity of the collar as it passes through Point $C$.



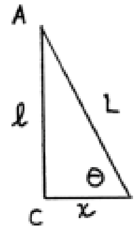

### SOLUTION

Choose the origin at Point $C$ and let $x$ be positive to the right. Then $x$ is a position coordinate of the slider $B$ and $x_0$ is its initial value. Let $L$ be the stretched length of the spring. Then, from the right triangle

$$L = \sqrt{\ell^2 + x^2}$$

The elongation of the spring is $e = L - \ell$, and the magnitude of the force exerted by the spring is

$$F_s = ke = k(\sqrt{\ell^2 + x^2} - \ell)$$

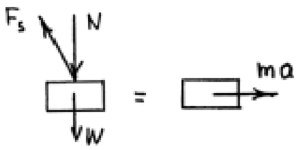



By geometry,
$$\cos\theta = \frac{x}{\sqrt{\ell^2 + x^2}}$$

$$\xrightarrow{+} \Sigma F_x = ma_x: \quad -F_s \cos\theta = ma$$

$$-k(\sqrt{\ell^2 + x^2} - \ell)\frac{x}{\sqrt{\ell^2 + x^2}} = ma$$

$$a = -\frac{k}{m}\left(x - \frac{\ell x}{\sqrt{\ell^2 + x^2}}\right)$$




$$\int_0^v v\,dv = \int_{x_0}^0 a\,dx$$

$$\frac{1}{2}v^2\Big|_0^v = -\frac{k}{m}\int_{x_0}^0\left(x - \frac{\ell x}{\sqrt{\ell^2 + x^2}}\right)dx = -\frac{k}{m}\left(\frac{1}{2}x^2 - \ell\sqrt{\ell^2 + x^2}\right)\Bigg|_{x_0}^0$$

$$\frac{1}{2}v^2 = -\frac{k}{m}\left(0 - \ell^2 - \frac{1}{2}x_0^2 + \ell\sqrt{\ell^2 + x_0^2}\right)$$

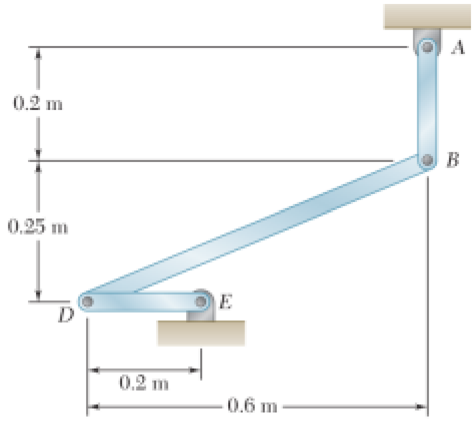
$$v^2 = \frac{k}{m}\left(2\ell^2 + x_0^2 - 2\ell\sqrt{\ell^2 + x_0^2}\right)$$

$$= \frac{k}{m}\left[\left(\ell^2 + x_0^2\right) - 2\ell\sqrt{\ell^2 + x_0^2} + \ell^2\right]$$

answer: $v = \sqrt{\frac{k}{m}}\left(\sqrt{\ell^2 + x_0^2} - \ell\right)$

## Ep 21.

Knowing that at the instant shown the angular velocity of rod *AB* is 15 rad/s clockwise, determine (*a*) the angular velocity of rod *BD*, (*b*) the velocity of the midpoint of rod *BD*.




### SOLUTION

Rod *AB*: $\boldsymbol{\omega}_{AB} = 15 \text{ rad/s}$ ↻

$v_B = (AB)\omega_{AB} = (0.200)(15) = 3 \text{ m/s}$ $\qquad \mathbf{v}_B = 3 \text{ m/s} \leftarrow$

Rod *BD*: $\mathbf{v}_B = -(3 \text{ m/s})\mathbf{i}, \quad \mathbf{v}_D = v_D\mathbf{j}, \quad \boldsymbol{\omega}_{BD} = \omega_{BD}\mathbf{k}$

$$\mathbf{r}_{B/D} = (0.6 \text{ m})\mathbf{i} + (0.25 \text{ m})\mathbf{j}$$

$$\mathbf{v}_B = \mathbf{v}_D + \mathbf{v}_{B/D} = \mathbf{v}_D + \boldsymbol{\omega}_{BD} \times \mathbf{r}_{B/D}$$

$$-3\mathbf{i} = v_D\mathbf{j} + \omega_{BD}\mathbf{k} \times (0.6\mathbf{i} + 0.25\mathbf{j})$$

$$= v_D\mathbf{j} + 0.6\omega_{BD}\mathbf{j} - 0.25\omega_{BD}\mathbf{i}$$

Equate components.

$$\mathbf{i}: \quad -3 = -0.25\omega_{BD} \qquad (1)$$

$$\mathbf{j}: \quad 0 = v_D + 0.6\omega_{BD} \qquad (2)$$

(*a*) *Angular velocity of rod BD.*

From Eq. (1), $\omega_{BD} = \dfrac{3}{0.25}$ $\qquad \boldsymbol{\omega}_{BD} = 12.00 \text{ rad/s}$ ↻

From Eq. (2), $v_D = -0.6\omega_{BD}$ $\qquad v_D = -7.2 \text{ m/s}$

(*b*) *Velocity of midpoint M of rod BD.*

$$\mathbf{r}_{M/D} = \frac{1}{2}\mathbf{r}_{B/D} = (0.3 \text{ m})\mathbf{i} + (0.125 \text{ m})\mathbf{j}$$

$$\mathbf{v}_M = \mathbf{v}_D + \mathbf{v}_{M/D} = v_D\mathbf{j} + \omega_{BD}\mathbf{k} \times \mathbf{r}_{M/D}$$

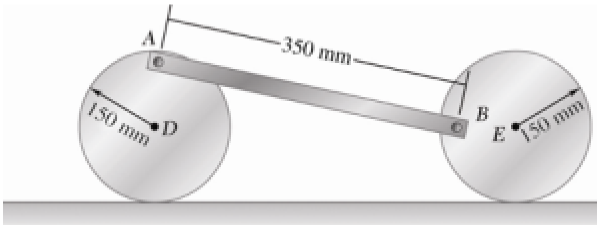
$$= -7.2\mathbf{j} + 12.00\mathbf{k} \times (0.3\mathbf{i} + 0.125\mathbf{j})$$

$$= -(1.500 \text{ m/s})\mathbf{i} - (3.60 \text{ m/s})\mathbf{j}$$

$$\mathbf{v}_M = 3.90 \text{ m/s} \measuredangle 67.4^\circ$$

## Ep 22.

Both 150 mm-radius wheels roll without slipping on the horizontal surface. Knowing that the distance $AD$ is 125 mm, the distance $BE$ is 100 mm and $D$ has a velocity of 150 mm/s to the right, determine the velocity of Point $E$.




**SOLUTION**

Disk $D$: Velocity at the contact Point $P$ with the ground is zero.

$$\mathbf{v}_0 = 150 \text{ mm/s} \longrightarrow$$

$$\omega_D = \frac{v_D}{r_{D/P}} = \frac{150 \text{ mm/s}}{150 \text{ mm}} = 1 \text{ rad/s} \qquad \boldsymbol{\omega}_D = 1 \text{ rad/s} \circlearrowright$$

At Point $A$,

$$v_A = r_{A/P}\omega_D = (150 \text{ mm} + 125 \text{ mm})(1 \text{ rad/s}) = 275 \text{ mm/s}$$

$$\mathbf{v}_A = 275 \text{ mm/s} \longrightarrow$$

Disk $E$: Velocity at the contact Point $Q$ with the ground is zero. $\boldsymbol{\omega}_E = \omega_E \circlearrowleft = \omega_E \mathbf{k}$.

$$\mathbf{r}_{B/Q} = -(100 \text{ mm})\mathbf{i} + (150 \text{ mm})\mathbf{j}$$

$$\mathbf{v}_B = \mathbf{v}_{B/Q} = \boldsymbol{\omega}_E \times \mathbf{r}_{B/Q} = \omega_E \mathbf{k} \times (-100\mathbf{i} + 150\mathbf{j})$$

$$\mathbf{v}_B = -150\omega_E \mathbf{i} - 100\omega_E \mathbf{j} \qquad (1)$$

Connecting rod $AB$:

$$\mathbf{r}_{B/A} = (\sqrt{(350)^2 - (125)^2})\mathbf{i} - 125\mathbf{j} \text{ in mm.}$$

$$\mathbf{v}_{B/A} = \sqrt{106875}\mathbf{i} - 125\mathbf{j} \qquad \boldsymbol{\omega}_{AB} = \omega_{AB}\mathbf{k}$$

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} = \mathbf{v}_A + \omega_{AB}\mathbf{k} \times (\sqrt{106875}\mathbf{i} - 125\mathbf{j})$$

$$= 275\mathbf{i} + 125\omega_{AB}\mathbf{i} + \sqrt{106875}\omega_{AB}\mathbf{j} \qquad (2)$$

Equating expressions (1) and (2) for $\mathbf{v}_B$ gives

$$-150\omega_E \mathbf{i} - 100\omega_E \mathbf{j} = 275\mathbf{i} + 125\omega_{AB}\mathbf{i} + \sqrt{106875}\omega_{AB}\mathbf{j}$$

Equating like components and transposing terms,

$\mathbf{i}$: $$125\omega_{AB} + 150\omega_E = -275 \qquad (3)$$

$\mathbf{j}$: $$\sqrt{106875}\omega_{AB} + 100\omega_E = 0 \qquad (4)$$

Solving the simultaneous equaitons (3) and (4),

$$\omega_{AB} = 0.75265 \text{ rad/s}, \quad \omega_E = -2.4605 \text{ rad/s}$$

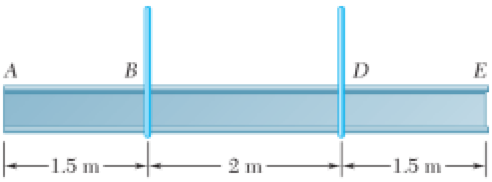
*Velocity of Point E.*

$$\mathbf{v}_E = \omega_E \mathbf{k} \times \mathbf{r}_{E/Q} = -2.4605\mathbf{k} \times 150\mathbf{j}$$

$$\mathbf{v}_E = 369 \text{ mm/s } \mathbf{i} = 369 \text{ mm/s} \longrightarrow$$

**Ep 23.**

For a 5-m steel beam *AE* the acceleration of point *A* is 2 m/s$^2$ downward and the angular acceleration of the beam is 1.2 rad/s$^2$ counterclockwise. Knowing that at the instant considered the angular velocity of the beam is zero, determine the acceleration (*a*) of cable *B*, (*b*) of cable *D*.

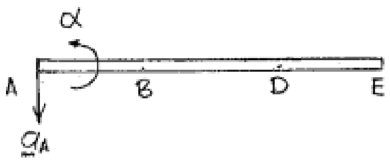



**SOLUTION**

$$\mathbf{a}_A = 2\ \text{m/s}^2 \downarrow, \qquad \boldsymbol{\alpha} = 1.2\ \text{rad/s}^2 \circlearrowleft$$

$$\mathbf{r}_{B/A} = 1.5\ \text{m} \longrightarrow, \qquad \mathbf{r}_{D/A} = 3.5\ \text{m} \longrightarrow$$

$$\boldsymbol{\omega} \approx 0$$




(*a*)

$$\mathbf{a}_B = \mathbf{a}_A + \left(\mathbf{a}_{B/A}\right)_t + \left(\mathbf{a}_{B/A}\right)_n$$

$$= [2\downarrow] + [(1.5)(1.2)\uparrow] + \left[(1.5)(0)^2 \longleftarrow\right]$$

$$= 0.2\ \text{m/s}^2 \downarrow$$

$$\mathbf{a}_B = 0.200\ \text{m/s}^2 \downarrow$$

(*b*)

$$\mathbf{a}_D = \mathbf{a}_A + \left(\mathbf{a}_{D/A}\right)_t + \left(\mathbf{a}_{D/A}\right)_n$$

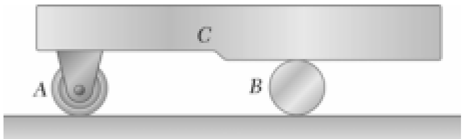
$$= [2\downarrow] + [(3.5)(1.2)\uparrow] + \left[(3.5)(0)^2 \longleftarrow\right]$$

$$= 2.2\ \text{m/s}^2 \uparrow$$

$$\mathbf{a}_D = 2.20\ \text{m/s}^2 \uparrow$$

## Ep 24.

A carriage $C$ is supported by a caster $A$ and a cylinder $B$, each of 50-mm diameter. Knowing that at the instant shown the carriage has an acceleration of $2.4\ \text{m/s}^2$ and a velocity of 1.5 m/s, both directed to the left, determine (*a*) the angular accelerations of the caster and of the cylinder, (*b*) the accelerations of the centers of the caster and of the cylinder.




### SOLUTION

Rolling occurs at all surfaces of contact. Instantaneous centers are at points of contact with floor.

Caster: $r = 0.025\ \text{m}$

$$\mathbf{a}_A = \mathbf{a}_C = 2.4\ \text{m/s}^2 \longleftarrow$$

$$(a_D)_x = 0 \text{ (rolling with no sliding)}$$

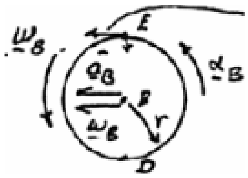
$$\mathbf{a}_A = \mathbf{a}_D + \mathbf{a}_{A/D}$$

$$[a_A \longleftarrow] = [(a_D)_x \longleftarrow] + [(a_D)_y \uparrow] + [rd_A \longleftarrow] + [r\omega_A^2 \downarrow]$$

$$\overset{+}{\longleftarrow}\ a_A = 0 + r\alpha_A$$

$$2.4\ \text{m/s}^2 \longleftarrow\ = (0.025\ \text{m})\alpha_A \quad \boldsymbol{\alpha}_A = 96\ \text{rad/s}^2 \circlearrowright$$

Cylinder: $r = 0.025\ \text{m}$



$$(\mathbf{a}_E)_x = \mathbf{a}_C = 2.4\ \text{m/s}^2 \longleftarrow$$

$$(a_E)_x = 0$$

$$\mathbf{a}_E = \mathbf{a}_D + \mathbf{a}_{E/D}$$

$$[(a_E)_x \longleftarrow] + [(a_E)_y \downarrow] = [(a_D)_x \longleftarrow] + [(a_D)_y \downarrow] + [2r\alpha_B \longleftarrow] + 2r\omega_B^2 \downarrow]$$

$$\overset{+}{\longleftarrow}:\quad (a_E)_x = (a_D)_y + 2r\alpha_B$$

$$[2.4\ \text{m/s}^2 \longleftarrow] = 0 + 2(0.025\ \text{m})\alpha_B$$

$$\boldsymbol{\alpha}_B = 48\ \text{rad/s}^2 \circlearrowright$$

$$[a_B \longleftarrow] = [(a_D)_x \longleftarrow] + [(a_D)_y \uparrow] + [r\alpha \longleftarrow] + [r\omega^2 \downarrow]$$

$$\overset{+}{\longleftarrow}:\quad a_B = 0 + r\alpha_B$$

$$a_B = (0.025\ \text{m})(48\ \text{rad/s}^2); \qquad \mathbf{a}_B = 1.2\ \text{m/s}^2 \longleftarrow$$

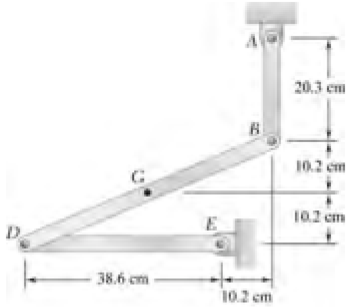
Answers:

(*a*) $\boldsymbol{\alpha}_A = 96.0\ \text{rad/s}^2 \circlearrowright, \quad \mathbf{a}_A = 2.40\ \text{m/s}^2$

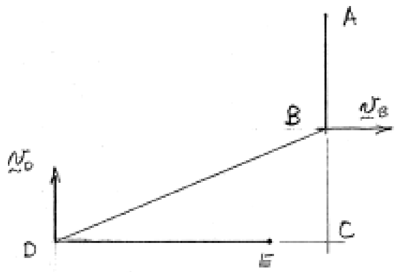
(*b*) $\boldsymbol{\alpha}_B = 48.0\ \text{rad/s}^2 \circlearrowright, \quad \mathbf{a}_B = 1.200\ \text{m/s}^2$

## Ep 25.

Knowing that at the instant shown bar *AB* has a constant angular velocity of 19 rad/s clockwise, determine (*a*) the angular acceleration of bar *BGD*, (*b*) the angular acceleration of bar *DE*.




**SOLUTION**




*Velocity analysis.* $\omega_{AB} = 19 \text{ rad/s} \circlearrowright$

$$v_B = (AB)\omega_{AB} = (.203)(19) = 3.857 \text{ m/s}$$

$$\mathbf{v}_B = v_B \longrightarrow, \quad \mathbf{v}_D = v_D \uparrow$$

Instantaneous center of bar *BD* lies at *C*.

$$\omega_{BD} = \frac{v_B}{BC} = \frac{3.875}{.203} = 19 \text{ rad/s} \circlearrowleft$$

$$v_D = (CD)\omega_{BD} = (.488)(19) = 9.27 \text{ m/s}$$

$$\omega_{DE} = \frac{v_D}{DE} = \frac{9.27}{.386} = 24 \text{ rad/s}^2 \circlearrowright$$

*Acceleration analysis.* $\alpha_{AB} = 0.$

$$\mathbf{a}_B = \left[(AB)\omega_{AB}^2 \uparrow\right] = \left[(.203)(19)^2 \uparrow\right] = 73.28 \text{ m/s}^2 \uparrow$$

$$\mathbf{a}_D = \left[(DE)\alpha_{DE} \downarrow\right] + \left[(DE)\omega_{DE}^2 \longrightarrow\right]$$

$$= \left[.386\alpha_{DE} \downarrow\right] + \left[222.3 \text{ m/s}^2 \longrightarrow\right]$$

$$\left(\mathbf{a}_{D/B}\right)_t = \left[.488\alpha_{BD} \downarrow\right] + \left[.203\alpha_{DB} \longrightarrow\right]$$

$$\left(\mathbf{a}_{D/B}\right)_n = \left[.488\omega_{BD}^2 \longrightarrow\right] + \left[.203\omega_{BD}^2 \uparrow\right]$$

$$= \left[176.168 \text{ m/s}^2 \longrightarrow\right] + \left[73.28 \text{ m/s}^2 \uparrow\right]$$

$$\mathbf{a}_D = \mathbf{a}_B + \left(\mathbf{a}_{D/B}\right)_t + \left(a_{D/B}\right)_n$$ Resolve into components.

$\overset{+}{\longrightarrow}$: $222.3 = 0 + .203\alpha_{BD} + 176.168$

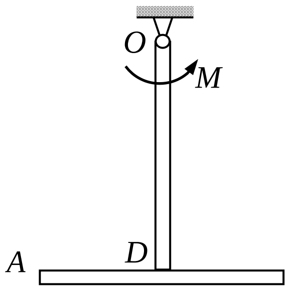
(*a*) $\alpha_{BD} = 227.25 \text{ rad/s}^2 \circlearrowleft$

$+\downarrow$: $.386\alpha_{DE} = -73.28 + (.488)(227.25) - 73.28$

(*b*) $\alpha_{DE} = -100 \text{ rad/s}^2$ $\qquad \alpha_{DE} = 100 \text{ rad/s}^2 \circlearrowright$

# Kinetic

**Ep 1:** Homogeneous rod length of $AB$ and $OD$ are $l$, quality are $m$, vertical consolidation into a T, and $D$ is the midpoint of $AB$ rod, placed in a vertical plane, as shown in figure (a), the t-shaped pole can turn round smooth horizontal axis, still at the beginning of the system, $OD$ pole of the vertical. Now the system turns under the action of couple $M = \frac{20}{\pi} mgl$. Try to find out the angular velocity, angular acceleration and the reaction force at bearing O of the T-bar when the $OD$ bar moves to the horizontal position.




Ep 1(a)

**SOLUTION:**

Take the T-bar as the research object, when it moves to the horizontal position of $OD$ bar, the force and movement are shown in the figure, and the coordinate as shown in the figure is established.

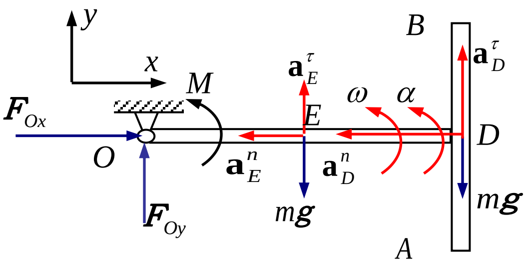
（1）Angular velocity $\omega$ is determined by the kinetic energy theorem.

The kinetic energy of the initial instantaneous system: $T_1 = 0$

When $OD$ rod reaches the horizontal position, the kinetic energy of the system is:

$$T_2 = \frac{1}{2} J_O \omega = \frac{1}{2}(\frac{1}{3}ml^2 + \frac{1}{12}ml^2 + ml^2)\omega^2$$

$$= \frac{17}{24}ml^2\omega^2$$




Ep 1(b)

The work done by all the forces in the process is:

$$\sum W_{12} = M \times \frac{\pi}{2} - mg\frac{l}{2} - mgl = \frac{17}{2}mgl$$

By the kinetic energy theorem for the system of particles $T_2 - T_1 = \sum W_{12}$, we have

$$\frac{17}{24}ml^2\omega^2 - 0 = \frac{17}{2}mgl$$

The angular velocity is: $\omega = \sqrt{\dfrac{12g}{l}} = 2\sqrt{\dfrac{3g}{l}}$ （Counterclockwise）

（2）Angular acceleration $\alpha$ is determined by the moment theorem (or the differential equation of fixed axis rotation).

The moment of momentum of T-bar to *O* axis is:

$$L_O = J_O\omega = \frac{17}{12}ml^2\omega$$

According to the moment of momentum theorem of particle system,

$$\frac{d}{dt}L_O = \sum m_O(\vec{F})$$

$$\frac{17}{12}ml^2\alpha = M - mg\frac{l}{2} - mgl$$

The angular velocity is: $\alpha = \dfrac{6g}{17\pi l}(40-3\pi)$ （Counterclockwise）

Of course, angular acceleration can also be obtained from the differential equation of fixed axis rotation $J_O\alpha = \sum m_O(\vec{F})$ .

（3）Use the law of motion of center of mass (or momentum theorem) to find the constrained reaction of bearing *O*.

By the center of mass motion theorem, $Ma_{Cx} = \sum F_x{}^e$ , $Ma_{Cy} = \sum F_y{}^e$

$$-ma_E^n - ma_D^n = F_{Ox}$$

$$ma_E^\tau + ma_D^\tau = F_{Oy} - 2mg$$

In which: $a_E^n = \dfrac{l}{2}\omega^2$ ; $a_E^\tau = \dfrac{l}{2}\alpha$ ; $a_D^n = l\omega^2$ ; $a_D^\tau = l\alpha$ , plug in the above equation, we get

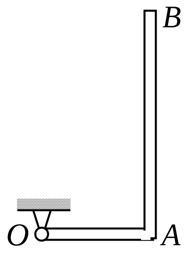
$$F_{Ox} = -\frac{3}{2}ml\omega^2 = -18mg$$

$$F_{Oy} = 2mg + \frac{3}{2}ml\alpha = 2mg + \frac{9mg}{17\pi}(40-3\pi)$$

Of course, it can also be solved by the momentum theorem.

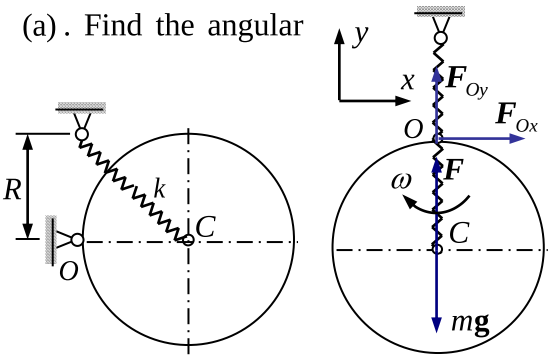
**Exercise:** The T-bar in the above example is only under the action of dead weight. Find out the angular velocity, angular acceleration and reaction force at bearing $O$ of T-bar when $OD$ bar rotates from horizontal position to vertical position from static position.

**Exercise:** The homogeneous rod $OA$ and $AB$, the length are $OA = l$, $AB = 2l$, total mass is $3m$, the bar is vertically consolidated and placed in the vertical plane of the lead, as shown in the figure. The bar can be rotated about a smooth horizontal axis. The system is stationary at the beginning and the $OA$ bar is horizontal. Try to find out the angular velocity, angular acceleration and reaction force at bearing $O$ of $OA$ bar when it turns to plumb position.



Ex 2

**Ep 2.** The mass of the homogeneous disk is $m$, the radius is $R$, the spring stiffness is $k$, and the original length is $R$. The disk is released without muzzle velocity at the position shown in Figure (a). Find the angular velocity, angular acceleration and the reaction force at bearing $O$ when the disc moves to the lowest position.



Ep 2(a)    Ep 2(b)

**SOLUTION:**

When the system moves to the position shown in Figure (b), the force and movement are shown in the figure, and the coordinates are established as shown in the figure.

（1）Calculate the angular velocity $\omega$.

The kinetic energy of the initial instantaneous system: $T_1 = 0$

When we go to the final state, the kinetic energy of the system is:

$$T_2 = \frac{1}{2} J_O \omega = \frac{1}{2}(\frac{1}{2} mR^2 + mR^2)\omega^2 = \frac{3}{4} mR^2\omega^2$$

The work done by all the forces in the process is:

$$\sum W_{12} = mgR + \frac{1}{2} k\left[(\sqrt{2}R - R)^2 - R^2\right] = mgR - \frac{1}{2} kR^2(2\sqrt{2} - 2)$$

By the kinetic energy theorem for the system of particles $T_2 - T_1 = \sum W_{12}$,we have

$$\frac{3}{4} mR^2\omega^2 - 0 = mgR - \frac{1}{2} kR^2(2\sqrt{2} - 2)$$

Solve it, we get $\omega^2 = \frac{4}{3}\left[\frac{g}{k} - \frac{k}{m}(\sqrt{2} - 1)\right]$ （Suppose $k$ is small enough, satisfy $\omega^2 > 0$ ）

Thus $$\omega = 2\sqrt{\frac{1}{3}\left[\frac{g}{R} - \frac{k}{m}(\sqrt{2} - 1)\right]}$$

（2）Calculate the angular acceleration $\alpha$ .

Since the external forces pass through the O-axis, so $\sum m_O(\vec{F}) = 0$, derived from the differential equation of the fixed axis rotation of the rigid body: $J_O\alpha = \sum m_O(\vec{F})$, we get $\alpha = 0$

（3）Find the constraint reaction of bearing *O*.

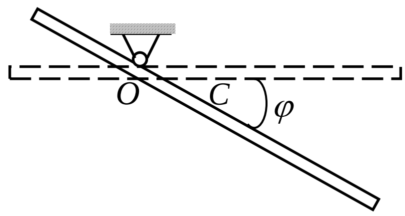
By the center of mass motion theorem $Ma_{Cx} = \sum F_x{}^e$， $Ma_{Cy} = \sum F_y{}^e$, we have

$$\begin{aligned} ma_{Cx} &= F_{Ox} \\ ma_{Cy} &= F_{Oy} + F - mg \end{aligned}$$

In which: $a_{Cx} = 0$； $a_{Cy} = R\omega^2$； $F = kR$ , plug in the above equation, we get

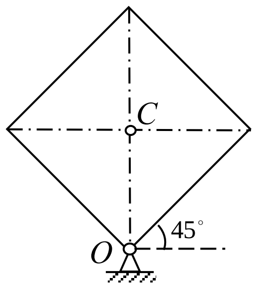
$$\begin{aligned} F_{Ox} &= 0 \\ F_{Oy} &= \frac{7}{3}mg - \frac{1}{3}kR(4\sqrt{2} - 1) \end{aligned}$$

**Exercise:** As shown in the figure, the homogeneous rod has a mass of $m$ and a length of $l$, and can be rotated around the rotation axis O of $l/3$ at the end point. Find out the angular velocity, angular acceleration and constraint reaction of bearing O when the bar rotates from horizontal position to any position from static position.

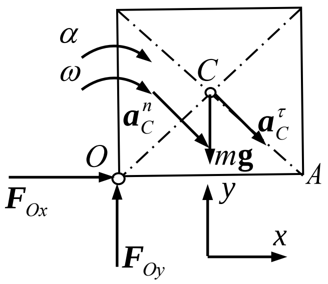


Ex 3

**Ep 3.** As shown in Figure (a), a homogeneous square plate with a mass of $m$ and a side length of $l$ is in a static state at the beginning. It falls clockwise after slight disturbance without any friction. Find the angular velocity, angular acceleration of the plate and the reaction force at bearing $O$ when the $OA$ side is in a horizontal position.




Ep 3(a)




Ep 3(b)

**SOLUTION:**

When the plate moves to the position shown in Figure (b), the force and movement are shown in the figure, and the coordinates are established as shown in the figure.

（1）Calculate the angular velocity $\omega$.

The kinetic energy of the initial instantaneous system: $T_1 = 0$

When in the final state, the kinetic energy of the system is:

$$T_2 = \frac{1}{2}J_O\omega^2 = \frac{1}{2}\left[\frac{1}{6}ml^2 + m(\frac{\sqrt{2}}{2}l)^2\right]\omega^2 = \frac{1}{3}ml^2\omega^2$$

The work done by all the forces in the process is:

$$\sum W_{12} = mg(\frac{\sqrt{2}}{2} - \frac{1}{2})l = \frac{1}{2}mgl(\sqrt{2}-1)$$

By the kinetic energy theorem for the system of particles $T_2 - T_1 = \sum W_{12}$, we have

$$\frac{1}{3}ml^2\omega^2 - 0 = \frac{1}{2}mgl(\sqrt{2}-1)$$

Obtain:
$$\omega = \sqrt{\frac{3g}{2l}(\sqrt{2}-1)}$$

（2）Calculate the angular acceleration $\alpha$ .

The differential equation for a fixed axis rotation by a rigid body is:

$$J_O\alpha = \sum m_O(\vec{F})$$

$$\frac{2}{3}ml^2\alpha = mg\frac{l}{2}$$

Thus:

$$\alpha = \frac{3g}{4l}$$

（3）Find the constraint reaction of bearing *O*.

By the center of mass motion theorem : $Ma_{Cx} = \sum F_x{}^e$, $Ma_{Cy} = \sum F_y{}^e$,

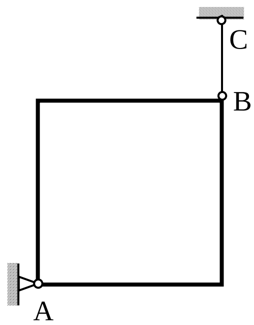
$$m(a_C^\tau \cos 45^\circ - a_C^n \cos 45^\circ) = F_{Ox}$$

$$m(-a_C^\tau \sin 45^\circ - a_C^n \sin 45^\circ) = F_{Oy} - mg$$

In which: $a_C^\tau = \frac{\sqrt{2}}{2}l\alpha$ ; $a_C^n = \frac{\sqrt{2}}{2}l\omega^2$, plug in the above equation, we get

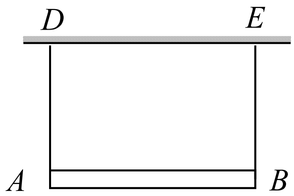
$$F_{Ox} = \frac{9-6\sqrt{2}}{8}mg \qquad F_{Oy} = \frac{11-6\sqrt{2}}{8}mg$$

**Exercise:** The homogeneous square sheet has a mass of m and is supported by hinge A and soft rope BC, as shown in the figure. When the plate rotates $90^\circ$ after the soft rope is cut, calculate the angular velocity, angular acceleration of the plate and the constrained reaction force of hinge A.



Ex 4

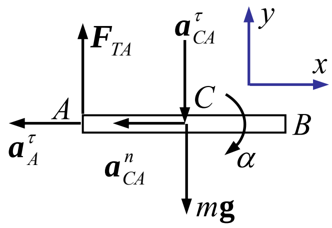
**Ep 4.** The homogeneous rod has a mass of $m$ and its two ends are suspended on two parallel lines. The rod is in a horizontal position, as shown in Figure (a). Suppose one of the ropes breaks suddenly, find the tension of the other rope at that moment.




Ep 4(a)

**SOLUTION:**

Take the bar as the research object, the force and movement are shown in Figure (b), and establish the coordinate as shown in the figure.By the differential equation of the plane motion of the rigid body, we have:



Ep 4(b)

$$ma_{Cx} = \sum F_x^e : \quad ma_{Cx} = 0 \qquad (1)$$

$$ma_{Cy} = \sum F_y^e : \quad ma_{Cy} = F_{TA} - mg \qquad (2)$$

$$J_C\varepsilon = \sum m_C(F) : \quad \frac{1}{12}ml^2\varepsilon = F_{TA}\frac{l}{2} \qquad (3)$$

At the moment the rope breaks, $\omega_{AB} = 0$, $v_A = 0$, $A$ only has tangential acceleration $\boldsymbol{a}_A^\tau$, The normal acceleration is $a_A^n = 0$, taking $A$ as the base point, the acceleration at point $C$ is analyzed as shown in Figure (b).

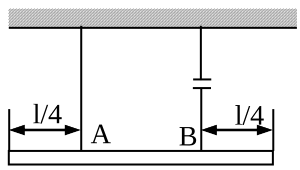
From: $\boldsymbol{a}_C = \boldsymbol{a}_A^\tau + \boldsymbol{a}_{CA}^\tau + \boldsymbol{a}_{CA}^n$ , project each vector onto the $y$-axis, and get

$$a_{Cy} = -a_{CA}^\tau = -\frac{l}{2}\varepsilon \qquad (4)$$

To solve the above four equations simultaneously, it can be obtained:

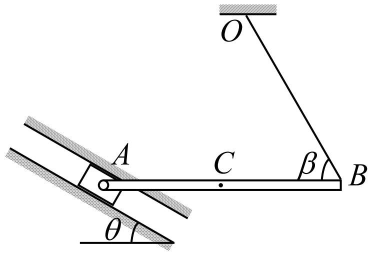
$$F_{TA} = \frac{1}{4}mg$$

**Exercise:** The homogeneous rod has a mass of m and a length of $l$, and it is suspended by two equal lengths of ropes as shown in the figure. Find the angular acceleration of the rod and the tension of the other rope when one rope is suddenly cut.




Ex 5

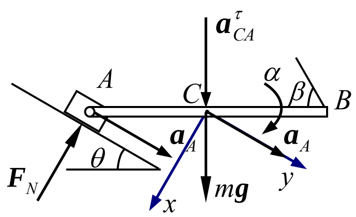
**Ep 5.** In the system shown in the figure, it is known that the mass of the homogeneous rod is $m$, $\theta = 30^\circ$, $\beta = 60^\circ$. Try to find the reaction force (excluding the mass of slider *A*) and the angular acceleration of bar *AB* when rope *OB* suddenly cuts the instantaneous chute.



Ep 5(a)

**SOLUTION:**

At the moment when the rope *OB* is cut, the angular velocity of the bar is zero, but the angular acceleration is not zero. The force and motion analysis of the bar AB are shown in Figure (b). Set up the graph coordinates. By the differential equation of the plane motion of the rigid body, we have:



Ep 5(b)

$$ma_{Cx} = mg\cos\theta - F_N \qquad (1)$$

$$ma_{Cy} = mg\sin\theta$$

$$\frac{1}{12}ml^2 \cdot \alpha = F_N\cos\theta \cdot \frac{l}{2} \qquad (2)$$

If A is the base point, then the acceleration at C is $\boldsymbol{a}_C = \boldsymbol{a}_A^{\tau} + \boldsymbol{a}_{CA}^{\tau} + \boldsymbol{a}_{CA}^{n}$,

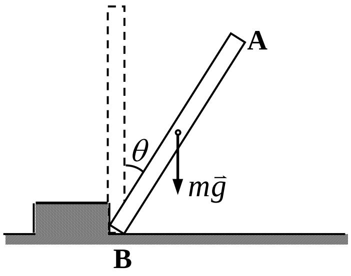
In which: $a_{CA}^{n} = 0$, project each vector onto the *x*-axis, and get

$$a_{Cx} = a_{CA}^{\tau}\cos\theta = \frac{l}{2}\alpha\cos\theta \qquad (3)$$

Simultaneous solutions (1), (2) and (3), it can be obtained as follows:

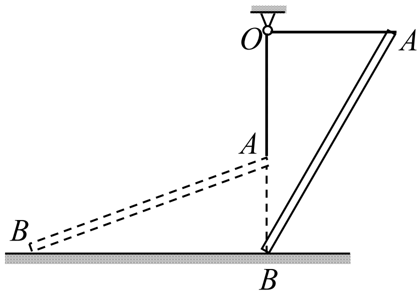
$$\alpha = \frac{6g\cos^2\theta}{l(1+3\cos^2\theta)} = \frac{18g}{13l} \qquad F_N = \frac{mg\cos\theta}{1+3\cos^2\theta} = \frac{2\sqrt{3}}{13}mg$$

**Exercise:** Homogeneous thin rod AB is $l$ in length and $m$ in mass. At first, the rod stands upright on a smooth plane with horizontal obstacles. Due to minor interference, the rod is tipped around point B as shown in the figure. Find out the angular velocity, angular acceleration of AB bar and the binding force at B when the B end is not detached from the obstacle.




Ex 6

**Ep 6.** As shown in the figure, the homogeneous rod has a mass of *m* and a length of *2l*. It is held on a smooth horizontal plane by *OA*, a thin rope of *l*. Figure out the velocity of point *B*, constraint reaction of the ground and tension of the rope when sliding from the position shown in the figure to the dotted line position.

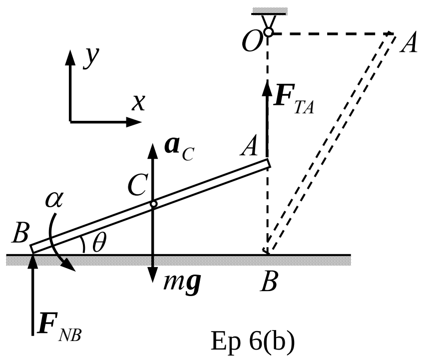



Ep 6(a)

**SOLUTION:**

（1）Calculate the velocity at point B by the kinetic energy theorem.

Rod AB was taken as the research object, and the force and motion analysis were shown in Figure (b). Set up the coordinates in graph.




Ep 6(b)

Since the system starts from static motion, the kinetic energy of the initial instantaneous system: $T_1 = 0$

When the AB bar moves to the dotted line position, it performs instantaneous translational action, which the kinetic energy of the system is:

$$T_2 = \frac{1}{2} m v_C^2$$

The work done by all the forces in the process is

$$\sum W_{12} = mg\left[\frac{\sqrt{3}}{2} l - \frac{1}{2}(\sqrt{3}l - l)\right] = \frac{1}{2} mgl$$

By the kinetic energy theorem for the system of particles, $T_2 - T_1 = \sum W_{12}$

$$\frac{1}{2} m v_C^2 - 0 = \frac{1}{2} mgl \quad , \quad v_C = \sqrt{gl}$$

$$\therefore v_B = v_C = \sqrt{gl}$$

（2）Find the constraint reaction of the ground and the tension of the rope.

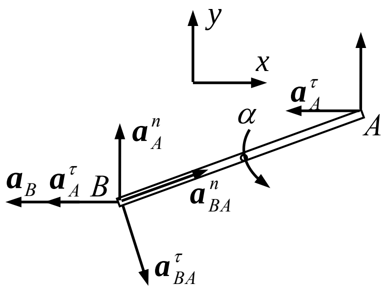
As the bar *AB* moves to the dotted line, $\sum F_x^e = 0$ , Therefore, the acceleration of the center of mass in the vertical direction is determined by the differential equation of motion in the plane of the rigid body:

$$F_{NB} + F_{TA} - mg = ma_C \qquad (1)$$

$$(F_{TA} - F_{NB})l\cos\theta = \frac{1}{12}m(2l)^2\alpha \qquad (2)$$

In equations (1) and (2), there are four unknowns, so kinematic analysis is required and supplementary equations are listed.

Taking $A$ as the base point, the resultant vector diagram of acceleration at Point $B$ is shown in Figure (c), where



Ep 6(c)

$$\boldsymbol{a}_B = \boldsymbol{a}_A^\tau + \boldsymbol{a}_A^n + \boldsymbol{a}_{BA}^\tau + \boldsymbol{a}_{BA}^n \qquad (3)$$

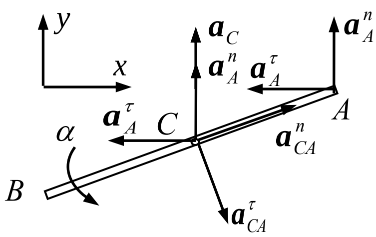
In which: $a_{BA}^n = 0$, $a_A^n = \dfrac{v_A^2}{l} = \dfrac{gl}{l} = g$, $a_{BA}^\tau = 2l\alpha$,

Project equation (3) onto the $y$-axis, and get

$$0 = a_A^n - a_{BA}^\tau\cos\theta \text{, which is } g - 2l\alpha\cos\theta = 0$$

Thus $$\alpha = \frac{g}{2l\cos\theta}$$

Then, taking $A$ as the base point, the resultant vector diagram of acceleration at point $C$ is shown in Figure (d), where



Ep 6(d)

$$\boldsymbol{a}_C = \boldsymbol{a}_A^\tau + \boldsymbol{a}_A^n + \boldsymbol{a}_{CA}^\tau + \boldsymbol{a}_{CA}^n \qquad (4)$$

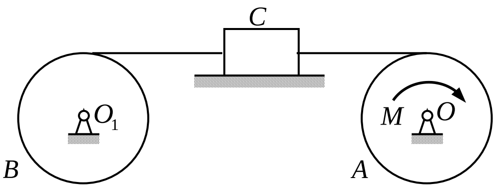
In which: $a_{CA}^n = 0$, $a_{CA}^\tau = l\alpha$, project equation (4) onto the y-axis, and get

$$a_C = a_A^n - a_{CA}^\tau\cos\theta = g - \frac{g}{2l\cos\theta}l\cos\theta = g - \frac{g}{2} = \frac{g}{2}$$

From the geometric relation, get $\cos\theta = \dfrac{\sqrt{2\sqrt{3}}}{2}$, plug $\alpha$, $a_C$ and $\cos\theta$ into equations (1), (2), and get

$$F_{TA} = \left(\frac{3}{4} + \frac{\sqrt{3}}{18}\right)mg \qquad F_{NB} = \left(\frac{3}{4} - \frac{\sqrt{3}}{18}\right)mg$$

**Ep 7.** As shown in the figure, wheels *A* and *B* can be regarded as homogeneous disks, with radius *R* and mass $m_1$. The rope around the two wheels is attached to block *C*. Let block *C* have a mass of $m_2$ and be placed on an ideal smooth horizontal plane. Apply a constant couple M on wheel *A*. Find the tension of the rope between wheel *A* and the block.



Ep 7(a)

**SOLUTION:**

Firstly take the system as the research object and use the kinetic energy theorem. Suppose the kinetic energy of an instantaneous system is $T_1$ (constant), when wheel *A* turns angle $\varphi$, the angular velocity is $\omega$, and the kinetic energy of the system is

$$T_2 = \frac{1}{2}J_O\omega^2 + \frac{1}{2}m_2(R\omega)^2 + \frac{1}{2}J_{O_1}\omega^2$$

In which: $J_O = J_{O_1} = \frac{1}{2}m_1R^2$, plug it in the equation above and we get

$$T_2 = \frac{1}{2}(m_1 + m_2)R^2\omega^2$$

The work done by all forces in motion: $\sum W = M\varphi$

By the kinetic energy theorem for the system of particles: $T_2 - T_1 = \sum W$ ,

We have， $\frac{1}{2}(m_1 + m_2)R^2\omega^2 - T_1 = M\varphi$

Take the derivative of both sides of this equation with respect to time,

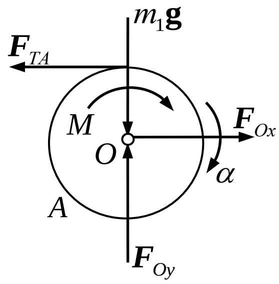
and notice that $\frac{d\varphi}{dt} = \omega$, $\frac{d\omega}{dt} = \alpha$ ,

then we have

$$(m_1 + m_2)R^2\omega\alpha = M\omega$$

$$\therefore \quad \alpha = \frac{M}{(m_1 + m_2)R^2}$$

Then take Wheel A as the research object, and the force is shown in Figure (b). The differential equation for a fixed axis rotation by a rigid body,



Ep 7(b)

$$J_O\alpha = M - F_{TA}R$$

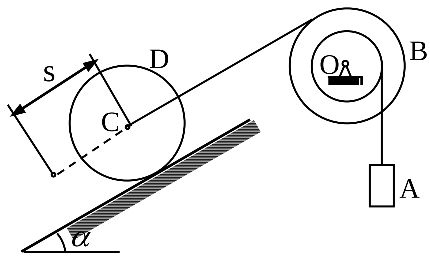
Which is

$$\frac{1}{2}m_1R^2\alpha = M - F_{TA}R$$

Plug $\alpha$ in, and we can get

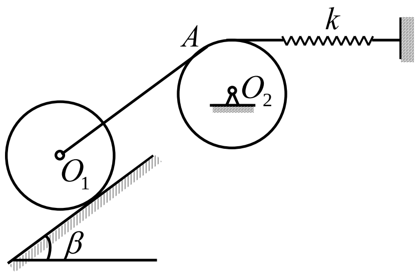
$$F_{TA} = \frac{M(m_1 + 2m_2)}{2R(m_1 + m_2)}$$

**Exercise:** The system in the figure, the mass of A is $m_1$, the mass of B is $m_2$, the radius of gyration of central axis is $\rho$, the radius of the wheel axis is $r$, the radius of the outer wheel is $R$, the mass of homogeneous disk D is $m_3$, and the radius is also R. It can make pure rolling along the inclined plane with an inclination angle of $\alpha$. Regardless of rope weight and friction at $O$-axis, the system is initially stationary. Try to find the velocity and acceleration of block A and the friction between the disk and the inclined plane when the disk center C moves downward along the inclined plane $s$.



Ex 7

**Ep 8.** As shown in the figure, the weights of homogeneous wheel $O_1$ and homogeneous wheel $O_2$ for pure rolling are $P$, the radius is $R$, the stiffness coefficient of the spring is $k$, and the inclination angle of the inclined plane is $\beta$. At the beginning, the system is stationary, and the spring is at the original length. There is no slip between the rope and the wheel $O_2$. The inclined section of the rope is parallel to the inclined plane, and the other section is horizontal. Calculate:




Ep 8(a)

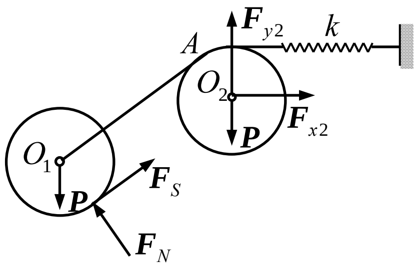
（1）The maximum distance the wheel $O_1$ can reach;

（2）The acceleration of the center of the wheel $O_1$ at this time;

（3）Tension in the $O_1A$ section of the rope.

**SOLUTION:**

（1）Taking the system as the research object, the force is shown in Figure (b).




Ep 8(b)

The kinetic energy of the initial instantaneous system $T_1 = 0$

If the maximum distance of wheel $O_1$ is $l$, then the kinetic energy of the final state system

$$T_2 = 0$$

The work done by all forces in motion:

$$\sum W = Pl\sin\beta - \frac{1}{2}kl^2$$

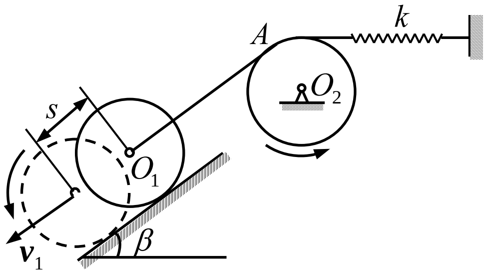
By the kinetic energy theorem for the system of particles $T_2 - T_1 = \sum W$,

$$0 = Pl\sin\beta - \frac{1}{2}kl^2$$

So: $$l = \frac{2P\sin\beta}{k}$$

（2）Taking the system as the research object, the kinetic energy of the initial instantaneous system is $T_1 = 0$

When the descent distance $s$ of wheel $O_1$ is set, the velocity and angular velocity of the wheel center are shown in figure (c), then the kinetic energy of the system is



Ep 8(c)

$$T_2 = \frac{1}{2}mv_1^2 + \frac{1}{2}J_1\omega_1^2 + \frac{1}{2}J_2\omega_2^2 = \frac{1}{2}\frac{P}{g}v_1^2 + \frac{1}{2}(\frac{1}{2}\frac{P}{g}R^2)\omega_1^2 + \frac{1}{2}(\frac{1}{2}\frac{P}{g}R^2)\omega_2^2$$

Due to $v_1 = R\omega_1 = R\omega_2$ , so

$$T_2 = \frac{1}{2}\frac{P}{g}v_1^2 + \frac{1}{4}\frac{P}{g}v_1^2 + \frac{1}{4}\frac{P}{g}v_1^2 = \frac{P}{g}v_1^2$$

The work done vigorously in the process of motion:

$$\sum W = Ps\sin\beta + \frac{1}{2}k(\delta_1^2 - \delta_2^2) = Ps\sin\beta - \frac{1}{2}ks^2$$

By the kinetic energy theorem for the system of particles $T_2 - T_1 = \sum W$, we have

$$\frac{P}{g}v_1^2 = Ps\sin\beta - \frac{1}{2}ks^2$$

The derivative of both sides of the above equation with respect to time, then:

$$2\frac{P}{g}v_1a_1 = P\sin\beta \cdot v_1 - \frac{1}{2}k \cdot 2sv_1$$

$$\therefore \quad a_1 = \frac{(P\sin\beta - ks)g}{2P}$$

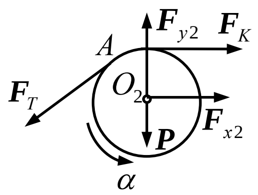
When $s = l, s = \dfrac{2P\sin\beta}{k}$, we get

$$a_1 = \frac{(P\sin\beta - k \cdot \frac{2P\sin\beta}{k})g}{2P} = -\frac{1}{2}g\sin\beta$$

（3）Find the tension of the rope $O_1A$.。

Take the wheel as the research object, and the force is shown in Figure (d). The differential equation of a fixed axis rotation by a rigid body is

$$J_2\alpha = F_T R - F_K R$$



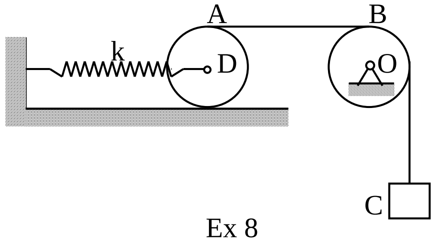
Ep 8(d)

In which: $J_2 = \frac{1}{2}\frac{P}{g}R^2, F_K = ks = kl$, plug it in, and:

$$\frac{1}{2}\frac{P}{g}R^2\alpha = (F_T - kl)R$$

$$F_T = \frac{1}{2}\frac{P}{g}R\alpha + kl = \frac{1}{2}\frac{P}{g}a_1 + kl = \frac{1}{2}\frac{P}{g}(-\frac{1}{2}g\sin\beta) + k\frac{2P\sin\beta}{k}$$
$$= -\frac{P}{4}\sin\beta + 2P\sin\beta = \frac{7}{4}P\sin\beta$$

**Exercise:** An inextensible string wound around roller A, over fixed pulley B and connected to block C of mass m at the other end. Fixed pulley B and roller A can be regarded as homogeneous disks with mass m and radius R. Roller A rolls purely on the horizontal plane, and the roller center is connected to A spring with A stiffness coefficient of k, as shown in the figure. Assuming that there is no relative sliding between the rope and the pulley, the friction at bearing O, the weight of the rope and spring are not taken into account. If the system is released at rest without deformation of the spring, block C begins to fall, try to calculate:



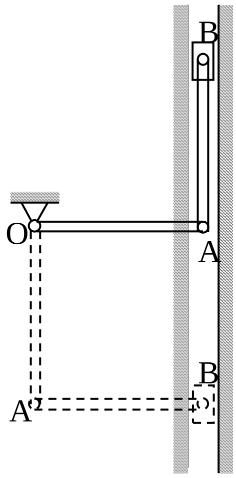
Ex 8

（1）The velocity and acceleration of the mass C at the descent distance h;

（2）Tension of AB rope;

（3）The friction between the horizontal plane and the roller.

**Ep 9.** The *OA* and *AB* hinge joints of two homogeneous rods of length *l* and weight *P* are shown. The slider *B* is confined to the vertical slot. If the system has no muzzle velocity release when the *OA* bar is in the horizontal position, try to find the velocities at Points *A* and *B* when the *AB* bar moves to the horizontal position, regardless of the friction and the mass of the slider *B*.



Ep 9

**SOLUTION:**

Taking the system as the research object, the kinetic energy of the initial instantaneous system is: $T_1 = 0$ .

When the system moves to the dotted line in the figure, the bar *AB* moves in a plane. The velocity at point *A* is in the horizontal direction, and the velocity at point *B* is in the vertical direction. Therefore, it can be determined that point *A* is the instantaneous center of velocity, so $v_A = 0$

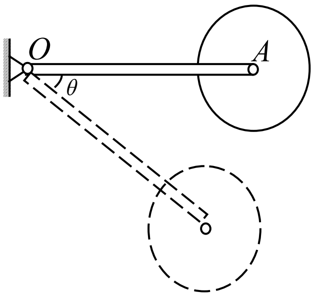
The kinetic energy of the system is: $T_2 = \frac{1}{2} J_A \omega_{AB}^2 = \frac{1}{2}(\frac{1}{3}\frac{P}{g}l^2)(\frac{v_B}{l})^2 = \frac{1}{6}\frac{P}{g}v_B^2$

The work done vigorously in the process of motion: $\sum W = P\frac{l}{2} + P\frac{3}{2}l = 2Pl$

By the kinetic energy theorem for the system of particles $T_2 - T_1 = \sum W$ , we have $\frac{1}{6}\frac{P}{g}v_B^2 - 0 = 2Pl$

Thus: $v_B = 2\sqrt{3gl}$

**Ep 10.** The homogeneous fine rod *OA* can rotate about the horizontal axis *O*, and the other end is hinged to a homogeneous disk, which can rotate freely around *A* in the lead plane, as shown in the figure. Given that bar *OA* is *l* in length, $m_1$ in mass, *R* in radius of disk, $m_2$ in mass, regardless of friction, bar *OA* is initially horizontal and bar and disk are stationary. Find the bar's angular velocity and angular acceleration at an angle $\theta$ between the bar and the horizontal line.



Ep 10(a)

**SOLUTION:**

Take the system as the research object. When the system moves to the position shown in the diagram, the force and movement are shown in Figure (b).

The initial instantaneous system is static, so the kinetic energy of the system is:

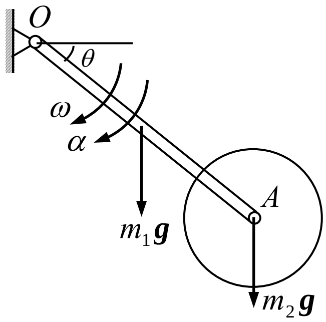
$$T_1 = 0$$

When the bar moves at an angle $\theta$ to the horizontal, due to the disk, the system's kinetic energy will be:

$$T_2 = \frac{1}{2}J_O\omega^2 + \frac{1}{2}m_2v_A^2$$

In which: $J_O = \frac{1}{3}ml^2, v_A = l\omega$, plug it in and:

$$T_2 = (\frac{1}{6}m_1 + \frac{1}{2}m_2)l^2\omega^2$$



Ep 10(b)

The work done vigorously in the process of motion:

$$\sum W = m_1 g\frac{l}{2}\sin\theta + m_2 gl\sin\theta$$

By the kinetic energy theorem for the system of particles $T_2 - T_1 = \sum W$,

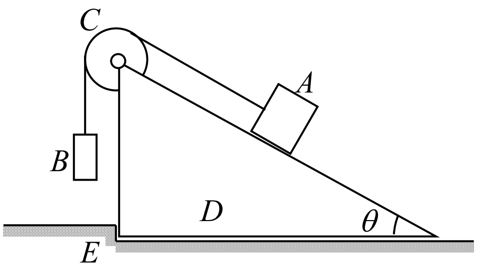
$$(\frac{1}{6}m_1 + \frac{1}{2}m_2)l^2\omega^2 - 0 = m_1 g\frac{l}{2}\sin\theta + m_2 gl\sin\theta$$

The angular velocity of the bar is $\omega = \sqrt{\dfrac{3g\sin\theta(m_1 + 2m_2)}{(m_1 + 3m_2)l}}$

The derivative with respect to time is the angular acceleration of the bar

$$\alpha = \frac{g\cos\theta(3m_1 + 6m_2)}{2l(m_1 + 3m_2)}$$

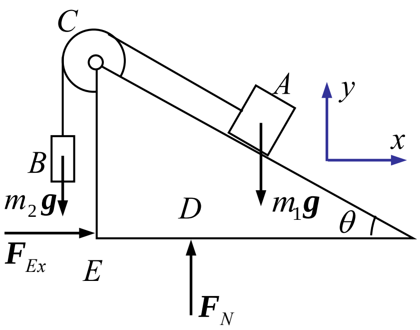
**Ep 11.** Block *A*, with mass $m_1$, slides down the slope of wedge *D*, while block *B*, with mass $m_2$, rises by a rope that bypasses pulley *C*, as shown in the figure. The inclined plane is $\theta$ angle to the horizontal, and the mass of the pulley and rope and any friction are ignored. Find the horizontal pressure of wedge *D* acting on convex part *E*.




Ep 11(a)

**SOLUTION:**

Taking the system as the research object, the force is shown in Figure (b), and the coordinate as shown in the figure is established. The kinetic energy theorem is applied.




Ep 11(b)

Let the kinetic energy of an instantaneous system be $T_1$(constant). When block *A* slides away from *s*, the velocities of *A* and *B* are both *v*, and then the kinetic energy of the system is:

$$T_2 = \frac{1}{2}(m_1 + m_2)v^2$$

The work done by all forces in motion: $\sum W = (m_1 g \sin\theta - m_2 g)s$

By the kinetic energy theorem for the system of particles: $T_2 - T_1 = \sum W$ ,

$$\frac{1}{2}(m_1 + m_2)v^2 - T_1 = (m_1 g \sin\theta - m_2 g)s$$

So take the derivative of both sides of this equation with respect to time, and we get this: $(m_1 + m_2)a = m_1 g \sin\theta - m_2 g$

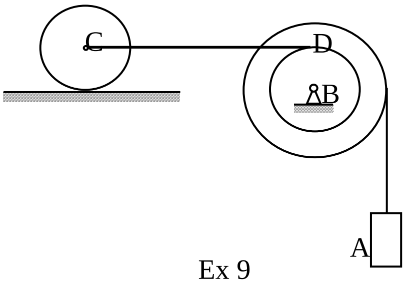
The acceleration of $A$ that we solve for is: $a = \dfrac{m_1 g \sin\theta - m_2 g}{m_1 + m_2}$

By the center of mass motion theorem $Ma_{Cx} = \sum F_x^e$ , $m_1 a \cos\theta = F_{Ex}$

We get:

$$F_{Ex} = \frac{m_1 \sin\theta - m_2}{m_1 + m_2} m_1 g \cos\theta$$

**Exercise:** In the diagram, it is known that the mass of block A is $m$, the mass of drum wheel B is $M$, the inner diameter is $r$, the outer diameter is $R$, the revolving radius of its central axis is $\rho$, the mass of homogeneous wheel is $m$, the radius is $r$, and pure rolling is carried out on the horizontal plane. The system starts at rest. Calculate:



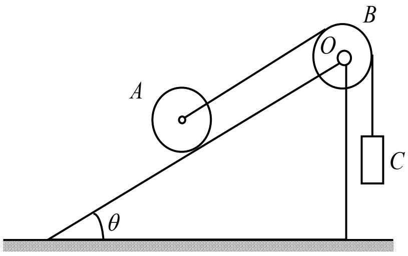
Ex 9

(1) The velocity and acceleration of the center of wheel C when the block A falls to the height of h;

(2) The tension of the horizontal rope CD;

(3) The horizontal friction acting on wheel C.

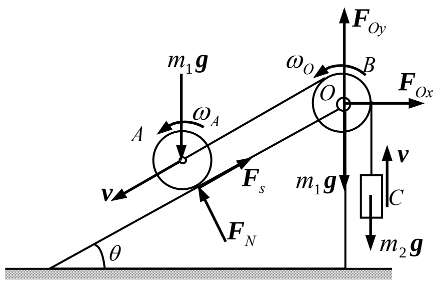
**Ep 12.** Roller *A* has a mass of $m_1$ and rolls down the inclined plane with an inclination of $\theta$ only without sliding, as shown in the figure. The roller lifts block *C* of $m_2$ by stepping over pulley *B*'s rope, while pulley *B* rotates around the *O*-axis. Roller *A* and pulley *B* have the same mass, the same radius, and are homogeneous disks. Find the acceleration of the center of gravity of the roller and the tension of the rope attached to the roller.




Ep 12(a)

**SOLUTION:**

Taking the system as the research object, the force and motion analysis are shown in the figure.




Ep 12(b)

Let the kinetic energy of an instantaneous system be $T_1$ (constant), when the downward motion distance of roller A is *s*, the kinetic energy of the system is: $T_2 = \frac{1}{2}J_{C^*}\omega_A^2 + \frac{1}{2}J_O\omega_O^2 + \frac{1}{2}m_2 v^2$

In which: $J_{C^*} = \frac{3}{2}m_1 r^2$, $J_O = \frac{1}{2}m_1 r^2$, $\omega_A r = \omega_O r = v$, plug in the equation above, we get: $T_2 = \frac{1}{2}(2m_1 + m_2)v^2$

The work done by all forces in motion: $\sum W = (m_1 g \sin\theta - m_2 g)s$

By the kinetic energy theorem for the system of particles: $T_2 - T_1 = \sum W$,

$$\frac{1}{2}(2m_1 + m_2)v^2 - T_1 = (m_1 g \sin\theta - m_2 g)s$$

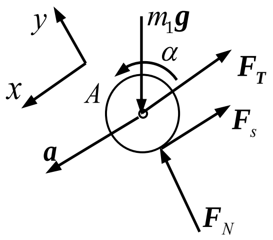
So take the derivative of both sides of this equation with respect to time, and get:

$$(2m_1 + m_2)a = m_1 g \sin\theta - m_2 g$$

So the acceleration of the center of mass of the roller A is: $a = \dfrac{m_1 g \sin\theta - m_2 g}{2m_1 + m_2}$

Taking roller A as the research object, the force and movement are shown in figure (c), and the coordinate as shown in the figure is established.

By the differential equation of the plane motion of the rigid body, we have




Ep 12(c)

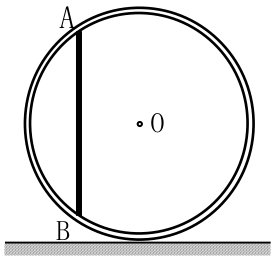
$$Ma_{Cx} = \sum F_x^e : \quad m_1 a = m_1 g \sin\theta - F_s - F_T$$

$$J_C\alpha = \sum m_C(\boldsymbol{F}) : \quad \frac{1}{2} m_1 r^2 \alpha = F_s r$$

Noticed that: $r\alpha = a$, solve the above two equations simultaneously, and get:

$$F_T = \frac{3m_1 m_2 + (2m_1 m_2 + m_1^2)\sin\theta}{2(2m_1 + m_2)} g$$

**Exercise:** As shown in the figure, the homogeneous ring has a radius of $r$ and a mass of $m$, which can be used for pure rolling on the horizontal plane. A rigid bar AB is welded on the ring, with a length of $\sqrt{3}r$ and a mass of $m$. At the beginning of the system, the AB bar is in the vertical position of the lead and starts to move from rest.




Ex 10

Calculate: (1) the angular acceleration of the ring at the beginning of the movement, the friction force of the ground facing the ring and the magnitude of the normal reaction;

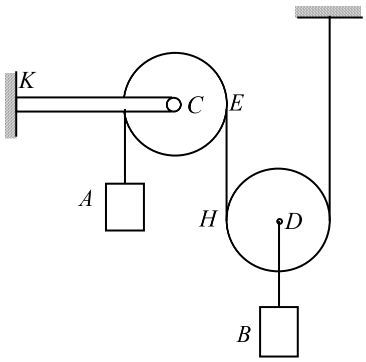
(2) The angular velocity of the ring when the system moves to the horizontal position of AB.

**Ep 13.** In the diagram, the mass of block *A* and *B* is $m$, the mass of both homogeneous round wheels *C* and *D* is $2m$, and the radius is *R*. Wheel *C* is hinged to *CK*, a weightless cantilever beam; *D* is movable pulley; the length of beam is 3*R*; there is no sliding between rope and wheel; the system starts from static motion. Figure out:

（1）The acceleration of block *A* rising;

（2）Tension of HE section rope;

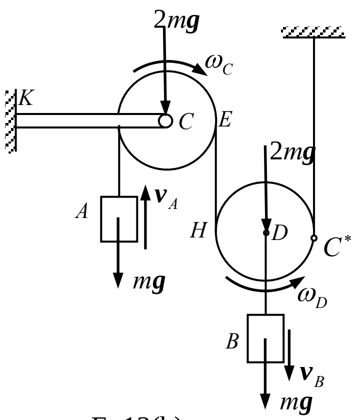
（3）Binding force at fixed end *K*.




Ep 13(a)

**SOLUTION:**

（1）Taking the system as the research object, the force and motion analysis are shown in Figure (b).




Ep13(b)

The initial instantaneous system is static, therefore $T_1 = 0$

When the distance in the above formula is $s$, the velocity is $\boldsymbol{v}_A$, then the drop of B is $\frac{s}{2}$, the velocity is $\frac{v_A}{2}$, and the instantaneous center of the wheel $D$ is at the point $C^*$. The kinetic energy of the system is:

$$T_2 = \frac{1}{2}mv_A^2 + \frac{1}{2}J_C\omega_C^2 + \frac{1}{2}mv_B^2 + \frac{1}{2}J_{C^*}\omega_D^2$$

In which: $\omega_C = \frac{v_A}{R}, v_B = \frac{v_A}{2}, \omega_D = \frac{v_A}{2R}, J_C = \frac{1}{2}\cdot 2mR^2, J_{C^*} = \frac{3}{2}\cdot 2mR^2$

Plug it in the equation above: $T_2 = \frac{3}{2}mv_A^2$

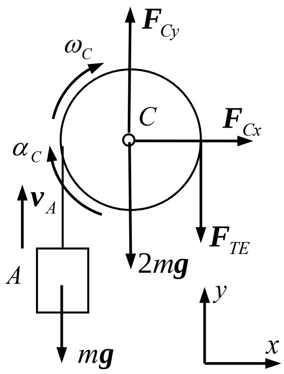
The work done by all forces in motion: $\sum W = -mgs + mg\frac{s}{2} + 2mg\frac{s}{2} = \frac{1}{2}mgs$

By the kinetic energy theorem for the system of particles: $T_2 - T_1 = \sum W$,

$$\frac{3}{2}mv_A^2 - 0 = \frac{1}{2}mgs$$

We take the derivative of both sides with respect to time, and we get that the acceleration of block $A$ going up is: $a_A = \frac{1}{6} g$

（2）The local system composed of block A and wheel C is taken as the research object, the force is shown in Figure (c), and the coordinates are established as shown in the figure.



Ep13(c

By the moment of momentum theorem $\frac{dL_C}{dt} = \sum m_C(\vec{F})$,

$$\frac{d}{dt}(J_C\omega_C + mv_A R) = F_{TE}R - mgR$$

In which: $\omega_C R = v_A, J_C = \frac{1}{2} \cdot 2mR^2$, plug it in and:

$$F_{TE} = \frac{4}{3} mg$$

By the center of mass motion theorem, we have:

$$Ma_{Cx} = \sum F_x^e : \quad 0 = F_{Cx}$$

$$Ma_{Cy} = \sum F_y^e : \quad ma_A = F_{Cy} - 2mg - mg - F_{TE}$$

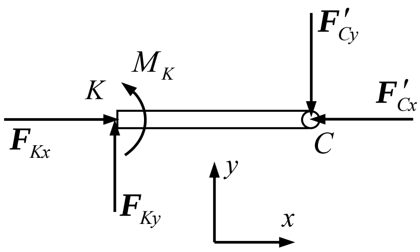
The binding force at hinge C can be solved as: $F_{Cx} = 0, \quad F_{Cy} = \frac{9}{2} mg$

（3）Taking the beam CK as the research object, the beam is in an equilibrium state and the stress is shown in Figure (d), and the coordinate as shown in the figure is established.

$$\sum F_x = 0 : F_{Kx} - F'_{Cx} = 0$$

$$\sum F_y = 0 : F_{Ky} - F'_{Cy} = 0$$

$$\sum m_K(\vec{F}) = 0 : M_K - 3R \cdot F'_{Cy} = 0$$

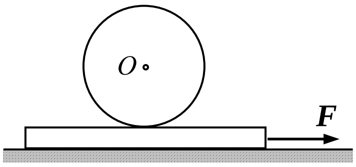


Ep 13(d)

The binding force of fixed end $K$ is obtained:

$$F_{Kx} = 0, \quad F_{Ky} = \frac{9}{2} mg, \quad M_K = \frac{27}{2} mgR$$

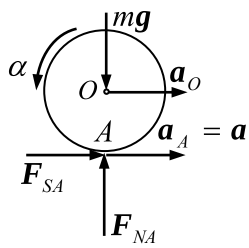
**Ep 14.** As shown in the figure, a metal plate with a mass of $M$ is placed on a smooth horizontal plane. There is a homogeneous cylinder with a radius of $R$ and a mass of $m$ on the plate, which can only make pure rolling on the plate. Now there is an ordinary force $\boldsymbol{F}$ pulling the metal plate. Calculate the acceleration of metal plate and angular acceleration of cylinder rolling.




Ep 14(a)

**SOLUTION:**

1. Using the rigid body plane motion differential equation, a cylinder is first taken as the research object. The force and motion analysis are shown in Figure (b). From the rigid body plane motion differential equation:




Ep 14(b)

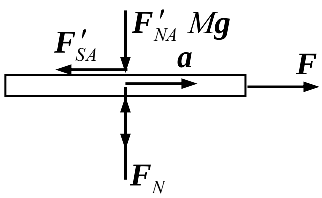
$$ma_O = F_{SA} \qquad (1)$$

$$(\frac{1}{2}mR^2)\alpha = F_{SA}R \qquad (2)$$

Then take the metal plate as the research object, and the force is shown in Figure (c). According to the motion theorem of the center of mass:

$$Ma = F - F'_{SA} \qquad (3)$$

Study the motion of a cylinder, take $A$ as the base point, then the acceleration at $O$ point is: $\boldsymbol{a}_O = \boldsymbol{a}_A + \boldsymbol{a}^{\tau}_{OA} + \boldsymbol{a}^{n}_{OA}$



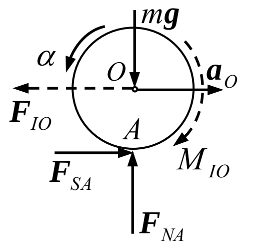

Ep 14(c)

Projection of the resultant acceleration vector equation to the horizontal direction: $a_O = a - R\alpha$ (4)

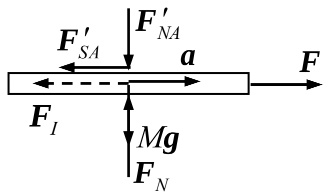
Because $F_{SA} = F'_{SA}$, Simultaneous solutions to equations (1) - (4), can obtained: $a = \dfrac{3F}{3M+m}$ $\quad \alpha = \dfrac{2F}{(3M+m)R}$

2. Application of D'Alembert's Principle (dynamic and static method)

Cylinder and metal plate are respectively taken as the research objects. A cylinder moves in a plane while a metal plate moves in a translational motion. Suppose inertial forces and inertial couple are added, as shown in Figure (d) (e).




Ep 14(d)




Ep 14(e)

In which: $F_{IO} = ma_O$ ,  $M_{IO} = J_O\alpha = (\frac{1}{2}mR^2)\alpha$ ,  $F_I = Ma$ , the direction is shown.

From D'Alembert's principle:

For a cylinder: $\sum F_x = 0$ :  $F_{SA} - F_{IO} = 0$  , $F_{SA} - ma_O = 0$  (5)

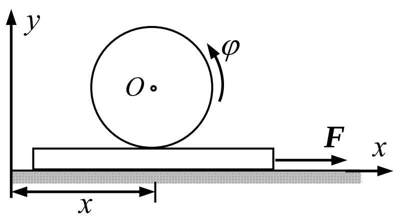
$\sum m_O(F) = 0$ :  $F_{SA}R - M_{IO} = 0$ ,  $F_{SA}R - (\frac{1}{2}mR^2)\alpha = 0$  (6)

For metal plates: $\sum F_x = 0$ : $F - F'_{SA} - F_I = 0$  ,  $F - F'_{SA} - Ma = 0$  (7)

Considering $F_{SA} = F'_{SA}$, the kinematics relation (4) and the equations (5) - (7) can be solved simultaneously, and the same result can be obtained.

Solution 3: Apply the Lagrange equation

Taking the system as the research object, the system has two degrees of freedom. $x$ and $\varphi$ are taken as generalized coordinates, as shown in Figure (f). The main force of the system is: $M\boldsymbol{g}$ 、 $m\boldsymbol{g}$ 、 $\boldsymbol{F}$ .




Ep 14(f)

Kinetic energy of the system:

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x} - R\dot{\varphi})^2 = \frac{1}{2}(\frac{1}{2}mR^2)\dot{\varphi}^2 = \frac{1}{2}(M+m)\dot{x}^2 - mR\dot{x}\dot{\varphi} + \frac{3}{4}mR^2\dot{\varphi}^2$$

Do the derivative:

$$\frac{\partial T}{\partial \dot{x}} = (M+m)\dot{x} - mR\dot{\varphi}\ ;\quad \frac{\partial T}{\partial x} = 0\ ;\quad \frac{d}{dt}(\frac{\partial T}{\partial \dot{x}}) = (M+m)\ddot{x} - mR\ddot{\varphi}$$

$$\frac{\partial T}{\partial \dot{\varphi}} = -mR\dot{x} + \frac{3}{2}mR^2\dot{\varphi}\ ;\quad \frac{\partial T}{\partial \varphi} = 0\ ;\quad \frac{d}{dt}(\frac{\partial T}{\partial \dot{\varphi}}) = -mR\ddot{x} + \frac{3}{2}mR^2\ddot{\varphi}$$

Calculate generalized force:

$$Q_x = \frac{\left[\sum \delta W\right]_x}{\delta x} = \frac{F\delta x}{\delta x} = F$$

$$Q_\varphi = \frac{\left[\sum \delta W\right]_\varphi}{\delta \varphi} = \frac{0}{\delta \varphi} = 0$$

By Lagrange's equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = Q_x\ , \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varphi}}\right) - \frac{\partial T}{\partial \varphi} = Q_\varphi\ ,$$

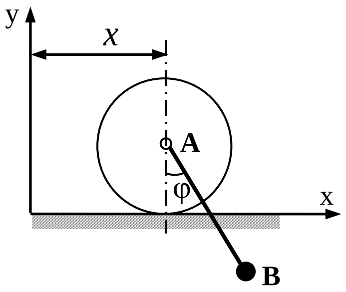
$$(M + m)\ddot{x} - mR\ddot{\varphi} = F \qquad (8)$$

$$-mR\ddot{x} + \frac{3}{2}mR^2\ddot{\varphi} = 0 \qquad (9)$$

Simultaneous solutions to equations (8) and (9), can obtain the same result:

$$\ddot{x} = a = \frac{3F}{3M + m} \qquad \ddot{\varphi} = \alpha = \frac{2F}{(3M + m)R}$$

**Exercise:** In the graphical system, it is known that cylinder A has a mass of M and a radius of $r$, and can be used for pure rolling on a fixed horizontal plane. The length of the single pendulum rod is $l$. The mass of the pendulum B is $m$, and the mass of the pendulum rod is excluded. Take $x$ and $\varphi$ as generalized coordinates.



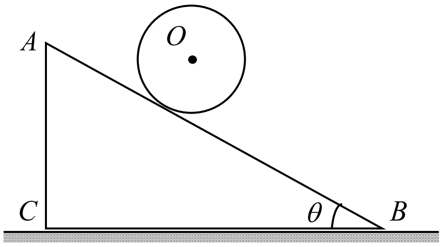
Ex 11

(1) Write down the kinetic energy of the system;

(2) Find the generalized force of the corresponding generalized coordinates (or write out the Lagrangian function);

(3) Use Lagrangian equation to find the differential equation of system motion.

**Ep 15.** The triangular prism ABC shown in the figure has a mass of *M*. Placed on a smooth horizontal surface, it can slide without friction. Homogeneous cylinder *O* with mass *m* and radius *r* rolls purely along the inclined plane *AB* from rest, if the inclination angle of the inclined plane is $\theta$. Given that the system is moving from rest, try to find the acceleration $\boldsymbol{a}$ of the triangular prism and the acceleration $\boldsymbol{a}_r$ of the center of the cylinder relative to the triangular prism.

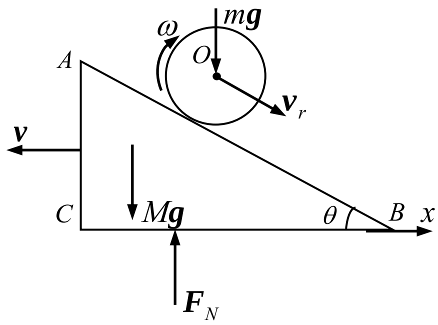



Ep 15(a)

**SOLUTION:**

1: Applying the general theorem of dynamics.

Taking the whole system as the research object, the force and motion analysis are shown in Figure (b), and the coordinate as shown in the figure is established.




Ep 15(b)

Because $\sum F_x^e = 0$, and the initial system is at rest , so $p_x = 0$.

Suppose the velocity of the triangular prism sliding to the left is $\boldsymbol{v}$, and the velocity of the cylindrical centroid *O* relative to the triangular prism is $\boldsymbol{v}_r$, then:

$$p_x = -Mv + m(v_r \cos\theta - v) = 0\text{, we get:} \quad v_r = \frac{M+m}{m\cos\theta}v \qquad (1)$$

Since the system is initially stationary, so $T_1 = 0$; The kinetic energy of the system when the cylinder rolls down the slope for distance s:

$$T_2 = \frac{1}{2}Mv^2 + \frac{1}{2}m(v^2 + v_r^2 + 2vv_r\cos\theta) + \frac{1}{2}(\frac{1}{2}mr^2)(\frac{v_r}{r})^2$$
$$= \frac{1}{2}Mv^2 + \frac{1}{2}m(v^2 + v_r^2 + 2vv_r\cos\theta) + \frac{1}{4}mv_r^2$$

Substitute Equation (1) into the above equation, and get:

$$T_2 = \frac{M+m}{4m\cos^2\theta}\left[3(M+m) - 2m\cos^2\theta\right]v^2 \qquad (2)$$

The work done by all forces in motion: $\sum W = mgs\sin\theta$

By the kinetic energy theorem for the system of particles, $T_2 - T_1 = \sum W$, we have: $\frac{M+m}{4m\cos^2\theta}\left[3(M+m) - 2m\cos^2\theta\right]v^2 = mgs\sin\theta$

Take the first derivative of both sides of the above equation with respect to time, and notice that $\frac{dv}{dt} = a$ and $\frac{ds}{dt} = v_r = \frac{M+m}{m\cos\theta}v$, can obtain:

$$a = \frac{mg\sin 2\theta}{3(M+m) - 2m\cos^2\theta} \qquad (3)$$

The first derivative of both sides of equation (1) with respect to time, can get

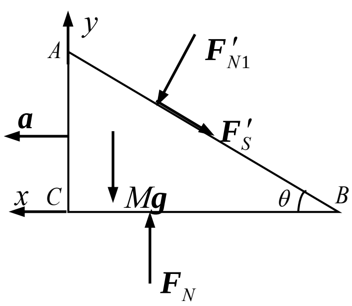
$$a_r = \frac{dv_r}{dt} = \frac{M+m}{m\cos\theta}a$$

Substitute $a$ into the above equation and we get:

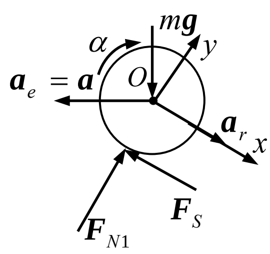
$$a_r = \frac{2(M+m)g\sin\theta}{3(M+m) - 2m\cos^2\theta} \qquad (4)$$

2: Apply the differential equation of motion in a rigid body plane

The triangular prism and cylinder are respectively taken as the research objects, and the forces are shown in (c) and (d), and the coordinates are established as shown in the figure.




Ep 15(c)




Ep 15(d)

For triangular prism: by the motion theorem of the center of mass

$$Ma = F'_{N1}\sin\theta - F'_S\cos\theta \qquad (5)$$

$$0 = F_N - Mg - F'_{N1} \cos\theta - F'_S \sin\theta \qquad (6)$$

For a cylinder: its center of mass does a point of resultant motion, the dynamic system is taken on a triangular prism, and the resultant theorem of acceleration with the involved motion as translational motion is as follows: $\boldsymbol{a}_O = \boldsymbol{a}_e + \boldsymbol{a}_r$, where the associated acceleration is $\boldsymbol{a}_e = \boldsymbol{a}$, and the relative acceleration is:

$$a_r = r\alpha \qquad (7)$$

Projection of the resultant acceleration vector $\boldsymbol{a}_O = \boldsymbol{a}_e + \boldsymbol{a}_r$ onto the $x$ and $y$ axes, and we obtained:

$$a_{Ox} = -a\cos\theta + a_r \qquad (8)$$

$$a_{Oy} = -a\sin\theta \qquad (9)$$

By the differential equation of the plane motion of the rigid body, we have

$$ma_{Ox} = mg\sin\theta - F_S \qquad (10)$$

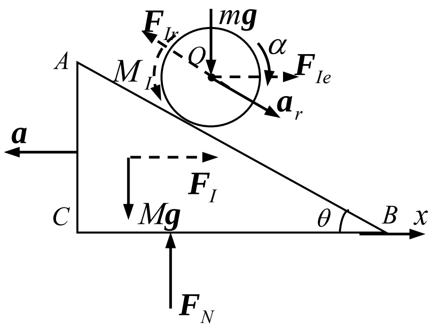
$$ma_{Oy} = F_{N1} - mg\cos\theta \qquad (11)$$

$$(\frac{1}{2}mr^2)\alpha = F_S r \qquad (12)$$

In which: $F'_{N1} = F_{N1}$, $F'_S = F_S$. Eight unknowns can be obtained by solving the eight equations (5) - (12) simultaneously. Where $a$ and $a_r$ are as shown in equations (3) and (4), get the same result.

3: Application of D 'Alembert's Principle (dynamic and static method)

Take the whole as the research object, the force is shown in Figure (e), and establish the coordinate as shown in the figure. The imaginary inertial force at the center of mass of the prism is $\boldsymbol{F}_I = -M\boldsymbol{a}$. The imaginary implicated inertia force on the cylinder center of mass $O$ is $\boldsymbol{F}_{Ie} = -m\boldsymbol{a}$ and the relative inertia



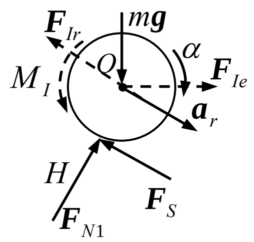

Ep 15(e)

force is $\boldsymbol{F}_{Ir} = -m\boldsymbol{a}_r$, and the virtual added moment of inertia is $M_I = -J_O\alpha$.

By D 'Alembert's principle: $\sum F_x = 0 : F_I + F_{Ie} - F_{Ir} \cos\theta = 0$

Which is $Ma + ma - ma_r \cos\theta = 0$, $a_r = \dfrac{M+m}{m\cos\theta} a$ (13)

Then take the cylinder as the research object, and the force is shown in Figure (f). On the cylinder center of mass $O$, the imaginary implicated inertia force is $\boldsymbol{F}_{Ie} = -m\boldsymbol{a}$, the relative inertia force is $\boldsymbol{F}_{Ir} = -m\boldsymbol{a}_r$, and the virtual added moment of inertia is $M_I = -J_O\alpha$.




Ep 15(f)

By D 'Alembert's principle:

$$\sum m_H(\boldsymbol{F}) = 0 : M_I + F_{Ir} r - F_{Ie} r\cos\theta - mgr\sin\theta = 0$$

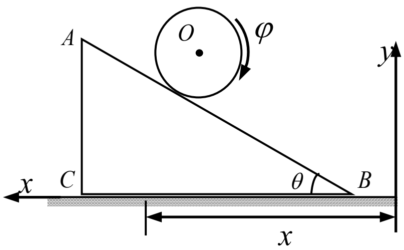
Which is: $(\frac{1}{2}mr^2)(\frac{a_r}{r}) + ma_r r - mar\cos\theta - mgr\sin\theta = 0$

$$\frac{3}{2}a_r = g\sin\theta + a\cos\theta \qquad (14)$$

To solve equations (13) and (14) simultaneously, the results are shown in Equations (3) and (4), and the same results are obtained.

4: Apply the Lagrange equation

Taking the system as the research object, the system has two degrees of freedom. $x$ and $\varphi$ are taken as generalized coordinates, as shown in Figure (g). The main forces of the system are $M\boldsymbol{g}$ and $m\boldsymbol{g}$.




Ep 15(g)

Kinetic energy of the system:

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + r^2\dot{\varphi}^2 - 2\dot{x}r\dot{\varphi}\cos\theta) + \frac{1}{2}(\frac{1}{2}mr^2)\dot{\varphi}^2$$

$$= \frac{1}{2}\left[(M+m)\dot{x}^2 + \frac{3}{2}mr^2\dot{\varphi}^2 - 2m\dot{x}r\dot{\varphi}\cos\theta\right]$$

Do the derivative:

$$\frac{\partial T}{\partial \dot{x}} = (M+m)\dot{x} - mr\dot{\varphi}\cos\theta\ ; \quad \frac{\partial T}{\partial x} = 0\ ; \quad \frac{d}{dt}(\frac{\partial T}{\partial \dot{x}}) = (M+m)\ddot{x} - mr\ddot{\varphi}\cos\theta$$

$$\frac{\partial T}{\partial \dot{\varphi}} = \frac{3}{2} mr^2 \dot{\varphi} - m\dot{x}r \cos\theta \text{ ; } \quad \frac{\partial T}{\partial \varphi} = 0 \text{ ; } \quad \frac{d}{dt}(\frac{\partial T}{\partial \dot{\varphi}}) = \frac{3}{2} mr^2 \ddot{\varphi} - m\ddot{x}r \cos\theta$$

Calculate the generalized force:

$$Q_x = \frac{\left[\sum \delta W\right]_x}{\delta x} = \frac{0}{\delta x} = 0$$

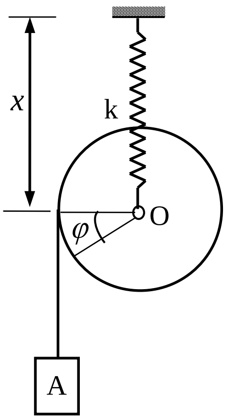
$$Q_\varphi = \frac{\left[\sum \delta W\right]_\varphi}{\delta \varphi} = \frac{mgr\delta\varphi \sin\theta}{\delta\varphi} = mgr \sin\theta$$

From the Lagrangian equation, $\frac{d}{dt}(\frac{\partial T}{\partial \dot{x}}) - \frac{\partial T}{\partial x} = Q_x$, $\frac{d}{dt}(\frac{\partial T}{\partial \dot{\varphi}}) - \frac{\partial T}{\partial \varphi} = Q_\varphi$, we get:

$$(M + m)\ddot{x} - mr\ddot{\varphi} \cos\theta = 0 \qquad (15)$$

$$\frac{3}{2} mr^2 \ddot{\varphi} - m\ddot{x}r \cos\theta = mgr \sin\theta \qquad (16)$$

Considering $\ddot{x} = a$ , $r\ddot{\varphi} = a_r$, to solve equations (15) and (16) simultaneously, the same result can be obtained.




Ex 12

**Exercise:** As shown in the figure, the wheel is a homogeneous disk with a mass of $m_1$ and a radius of R. The wheel center O and the weight A can only move along the straight direction. The mass of the weight A is $m_2$, the spring stiffness coefficient is $k$, and the original length is $l_0$. Take $x$ and $\varphi$ as generalized coordinates.

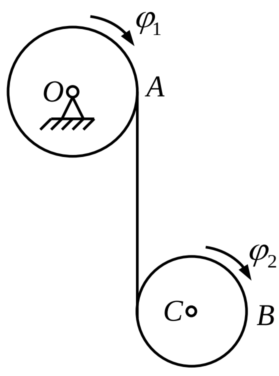
(1) Write down the kinetic energy of the system;

(2) Find the generalized forces corresponding to the generalized coordinates;

(3) Try to find the differential equation of motion of the system.

**Exercise:** As shown in the figure, the mass of homogeneous cylinder $A$ and $B$ is $m_1$ and $m_2$, and the radius is $r_1$ and $r_2$, respectively. Cylinder $A$ may rotate about a fixed axis $O$. One rope is wrapped around cylinder $A$ and the other end of the rope is wrapped around cylinder $B$. The mass of the string and the friction of the bearing are neglected. Calculate:

(1) Write the kinetic energy of the system with $\varphi_1$ and $\varphi_2$ as the generalized coordinates;




Ex 13

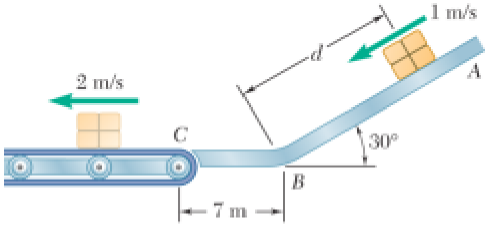
(2) Find the generalized forces corresponding to the generalized coordinates;

(3) Lagrange equation is used to establish the differential equation of motion of the system;

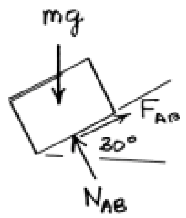
(4) Find the angular acceleration of the two cylinders and the acceleration of the center of mass *C* of cylinder *B*.

**Ep 16.**

Packages are thrown down an incline at $A$ with a velocity of 1 m/s. The packages slide along the surface $ABC$ to a conveyor belt which moves with a velocity of 2 m/s. Knowing that $\mu_k = 0.25$ between the packages and the surface $ABC$,determine the distance $d$ if the packages are to arrive at $C$ with a velocity of 2 m/s.




## SOLUTION

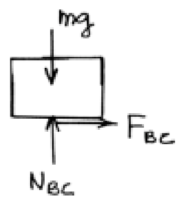



On incline $AB$:

$$N_{AB} = mg\cos 30°$$
$$F_{AB} = \mu_k N_{AB} = 0.25mg\ \cos 30°$$
$$U_{A\to B} = mgd\sin 30° - F_{AB}\,d$$
$$= mgd(\sin 30° - \mu_k \cos 30°)$$

On level surface $BC$:

$$N_{BC} = mg \qquad x_{BC} = 7\text{ m}$$
$$F_{BC} = \mu_k mg$$
$$U_{B\to C} = -\mu_k mg\ x_{BC}$$




At $A$,

$$T_A = \frac{1}{2}mv_A^2 \quad \text{and} \quad v_A = 1\text{ m/s}$$

At $C$,

$$T_C = \frac{1}{2}mv_C^2 \quad \text{and} \quad v_C = 2\text{ m/s}$$

Assume that no energy is lost at the corner $B$.

Work and energy.

$$T_A + U_{A\to B} + U_{B\to C} = T_C$$

$$\frac{1}{2}mv_A^2 + mgd(\sin 30° - \mu_k \cos 30°) - \mu_k mg\,x_{BC} = \frac{1}{2}mv_0^2$$

Dividing by $m$ and solving for $d$,

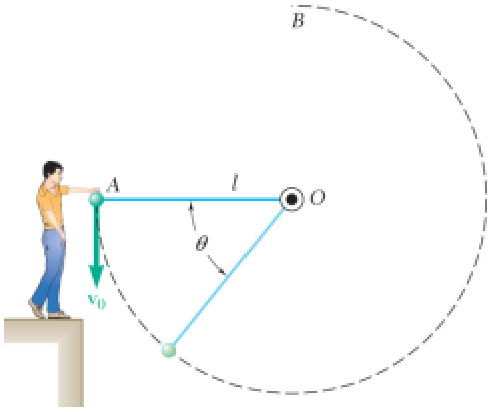
$$d = \frac{\left[v_C^2/2g + \mu_k x_{BC} - v_A^2/2g\right]}{(\sin 30° - \mu_k \cos 30°)}$$

$$= \frac{(2)^2/(2)(9.81) + (0.25)(7) - (1)^2/(2)(9.81)}{\sin 30° - 0.25\cos 30°}$$

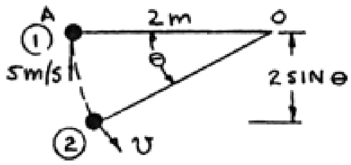
$d = 6.71$ m

## Ep 17.

The sphere at $A$ is given a downward velocity $\mathbf{v}_0$ of magnitude 5 m/s and swings in a vertical plane at the end of a rope of length $l = 2$ m attached to a support at $O$. Determine the angle $\theta$ at which the rope will break, knowing that it can withstand a maximum tension equal to twice the weight of the sphere.




**SOLUTION**




$$T_1 = \frac{1}{2}mv_0^2 = \frac{1}{2}m\,(5)^2$$

$$T_1 = 12.5\text{ m}$$

$$T_2 = \frac{1}{2}mv^2$$

$$U_{1-2} = mg(l)\sin\theta$$

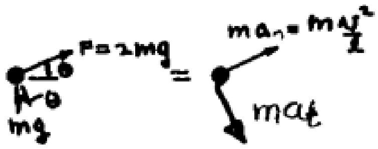
$$T_1 + U_{1-2} = T_2 \qquad 12.5m + 2mg\sin\theta = \frac{1}{2}mv^2$$

$$25 + 4g\sin\theta = v^2 \qquad (1)$$

*Newton's law at②.*

$$+\nearrow\ 2mg - mg\sin\theta = m\frac{v^2}{\ell} = m\frac{v^2}{2} \qquad (2)$$

$$v^2 = 4g - 2g\sin\theta$$



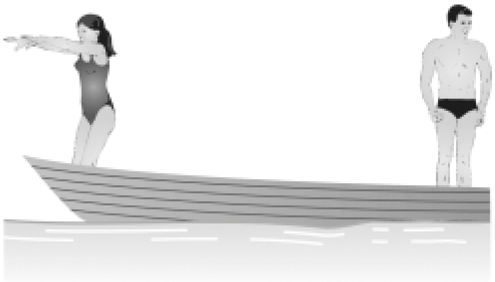

Substitute for $v^2$ from Eq. (2) into Eq. (1)

$$25 + 4g\sin\theta = 4g - 2g\sin\theta$$

$$\sin\theta = \frac{(4)(9.81) - 25}{(6)(9.81)} = 0.2419 \qquad \theta = 14.00°$$

**Ep 18.**

A 90-kg man and a 60-kg woman stand at opposite ends of a 150-kg boat, ready to dive, each with a 5-m/s velocity relative to the boat. Determine the velocity of the boat after they have both dived, if (*a*) the woman dives first, (*b*) the man dives first.



**SOLUTION**

(*a*) Woman dives first.

Conservation of momentum:


$$-60(5 - v_1) + 240v_1 = 0$$

$$v_1 = \frac{300}{300} = 1 \text{ m/s} \longrightarrow$$

Man dives next. Conservation of momentum:


$$240v_1 = -150v_2 + 90(5 - v_2)$$

$$v_2 = \frac{450 + 240v_1}{240} = 0.875 \text{ m/s} \qquad \mathbf{v}_2 = 0.875 \text{ m/s} \longleftarrow$$

(*b*) Man dives first.

Conservation of momentum:

$$90(5 - v_1') - 210v_1' = 0$$

$$v_1' = \frac{450}{300} = 1.5 \text{ m/s} \longleftarrow$$

Woman dives next. Conservation of momentum:

$$-210v_1' = 150v_2' - 60(5 - v_2')$$

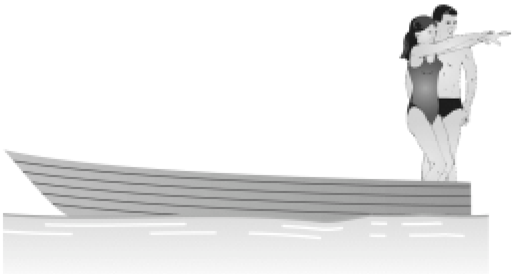
$$v_2' = -210v_1' = 210v_2' - 300$$

$$v_2' = \frac{-210v_1' + 300}{210} = \frac{-210 \times 1.5 + 300}{210} = -0.0714 \text{ m/s}$$

$$\mathbf{v}_2' = -0.0714 \text{ m/s} \longleftarrow$$

**Ep 19.**

A 90-kg man and a 60 kg woman stand side by side at the same end of a 150-kg boat ready to dive, each with a 5-m/s velocity relative to the boat. Determine the velocity of the boat after they have both dived, if (*a*) the woman dives first, (*b*) the man dives first.



**SOLUTION**

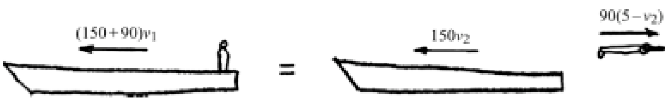
(*a*) Woman dives first.

Conservation of momentum:


$$60(5 - v_1) - (150 + 90)v_1 = 0$$

$$v_1 = \frac{300}{300} = 1 \text{ m/s} \longleftarrow$$

Man dives next. Conservation of momentum:




$$-240v_1 = -150v_2 + 90(5 - v_2)$$

$$v_2 = \frac{240v_1 + 450}{240} = 2.875 \text{ m/s} \qquad \mathbf{v}_2 = 2.88 \text{ m/s} \longleftarrow$$

(*b*) Man dives first.

Conservation of momentum:

$$90(5 - v_1') - 210v_1' = 0$$

$$v_1' = \frac{450}{300} = 1.5 \text{ m/s}$$

Woman dives next. Conservation of momentum:

$$-210v_1' = -150v_2' + 60(5 - v_2')$$

$$v_2' = \frac{210v_1' + 300}{210} = 2.9286 \text{ m/s} \qquad \mathbf{v}_2' = 2.93 \text{ m/s} \longleftarrow$$

## Ep 20.

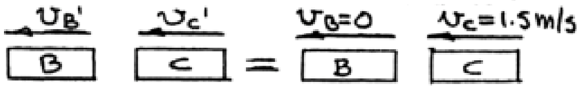
Two identical cars $A$ and $B$ are at rest on a loading dock with brakes released. Car $C$, of a slightly different style but of the same weight, has been pushed by dockworkers and hits car $B$ with a velocity of 1.5 m/s. Knowing that the coefficient of restitution is 0.8 between $B$ and $C$ and 0.5 between $A$ and $B$, determine the velocity of each car after all collisions have taken place.


### SOLUTION

$$m_A = m_B = m_C = m$$

Collision between $B$ and $C$:

The total momentum is conserved:




$$\overset{+}{\leftarrow} \quad mv'_B + mv'_C = mv_B + mv_C$$

$$v'_B + v'_C = 0 + 1.5 \qquad (1)$$

Relative velocities:

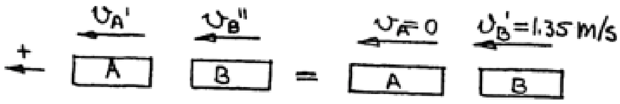
$$(v_B - v_C)(e_{BC}) = (v'_C - v'_B)$$
$$(-1.5)(0.8) = (v'_C - v'_B)$$
$$-1.2 = v'_C - v'_B \qquad (2)$$

Solving (1) and (2) simultaneously,

$$v'_B = 1.35 \text{ m/s}$$
$$v'_C = 0.15 \text{ m/s} \qquad \mathbf{v}'_C = 0.150 \text{ m/s} \longleftarrow$$

Since $v'_B > v'_C$, car $B$ collides with car $A$.

Collision between$A$and$B$:




$$mv'_A + mv''_B = mv_A + mv'_B$$
$$v'_A + v''_B = 0 + 1.35 \qquad (3)$$

Relative velocities:

$$(v_A - v'_B)e_{AB} = (v''_B - v'_A)$$
$$(0 - 1.35)(0.5) = v''_B - v'_A$$
$$v'_A - v''_B = 0.675 \qquad (4)$$

Solving (3) and (4) simultaneously,

$$2v'_A = 1.35 + 0.675$$

$$\mathbf{v}'_A = 1.013 \text{ m/s} -$$

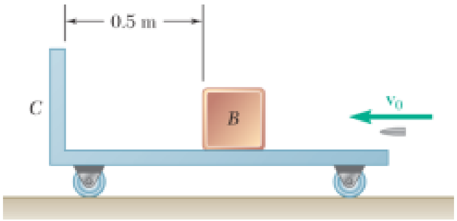
$$\mathbf{v}''_B = 0.338 \text{ m/s} -$$

Since $v'_C < v''_B < v'_A$, there are no further collisions.

**Ep 21.**

Determine the energy lost due to friction and the impacts

A 30-g bullet is fired with a horizontal velocity of 450 m/s and becomes embedded in block *B* which has a mass of 3 kg. After the impact, block *B* slides on 30-kg carrier *C* until it impacts the end of the carrier. Knowing the impact between *B* and *C* is perfectly plastic and the coefficient of kinetic friction between *B* and *C* is 0.2, determine (*a*) the velocity of the bullet and *B* after the first impact, (*b*) the final velocity of the carrier.




## SOLUTION

From the solution to Problem 4.1 the velocity of *A* and *B* after the first impact is $v' = 4.4554$ m/s and th velocity common to *A*, *B*, and *C* after the sliding of block *B* and bullet *A* relative to the carrier *C* has ceased i $v'' = 0.4087$ m/s.

Friction loss due to sliding:

Normal force: $N = W_A + W_B = (m_A + m_B)g$

$= (0.030 \text{ kg} + 3 \text{ kg})(9.81 \text{ m/s}) = 29.724 \text{ N}$

Friction force: $F_f = \mu_k N = (0.2)(29.724) = 5.945 \text{ N}$

Relative sliding distance: Assume $d = 0.5$ m.

Energy loss due to friction: $F_f d = (5.945)(0.5)$ $\qquad F_f d = 2.97 \text{ J}$

Kinetic energy of block with embedded bullet immediately after first impact:

$$T'_{AB} = \frac{1}{2}(m_A + m_B)(v')^2 = \frac{1}{2}(3.03 \text{ kg})(4.4554 \text{ m/s})^2 = 30.07 \text{ J}$$

Final kinetic energy of *A*, *B*, and *C* together

$$T''_{ABC} = \frac{1}{2}(m_A + m_B + m_C)(v'')^2 = \frac{1}{2}(33.03 \text{ kg})(0.4087 \text{ m/s})^2 = 2.76 \text{ J}$$

Loss due to friction and stopping impact: $T'_{AB} - T''_{ABC} = 30.07 - 2.76 = 27.31 \text{ J}$

Since $27.31 \text{ J} \geq 2.97 \text{ J}$, the block slides 0.5 m relative to the carrier as assumed above.

Impact loss due to *AB* impacting the carrier: $27.31 - 2.97 = 24.34$ $\qquad$ Loss $= 24.3$ J

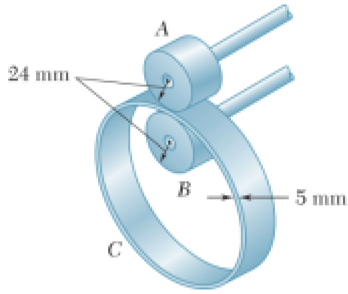
Initial kinetic energy of system *ABC*.

$$T_0 = \frac{1}{2}m_A v_0^2 = \frac{1}{2}(0.030 \text{ kg})(450 \text{ m/s})^2 = 3037.5 \text{ J}$$

Impact loss at first impact: $T_0 - T'_{AB} = 3037.5 - 30.07$ $\qquad$ Loss $= 3007$ J

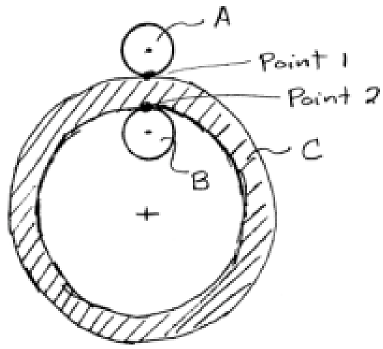
**Ep 22.**

Ring $C$ has an inside radius of 55 mm and an outside radius of 60 mm and is positioned between two wheels $A$ and $B$, each of 24-mm outside radius. Knowing that wheel $A$ rotates with a constant angular velocity of 300 rpm and that no slipping occurs, determine ($a$) the angular velocity of the ring $C$ and of wheel $B$, ($b$) the acceleration of the Points on$A$ and $B$that are in contact with $C$.




**SOLUTION**

$$\omega_A = 300 \text{ rpm}\left(\frac{2\pi}{60}\right)$$
$$= 31.416 \text{ rad/s}$$
$$r_A = 24 \text{ mm}$$
$$r_B = 24 \text{ mm}$$
$$r_1 = 60 \text{ mm}$$
$$r_2 = 55 \text{ mm}$$




[We assume senses of rotation shown for our computations.]

($a$) Velocities:

Point 1 (Point of contact of $A$ and $C$)

$$v_1 = r_A\omega_A = r_1\omega_C$$
$$\omega_C = \frac{r_A}{r_1}\omega_A$$
$$= \frac{24 \text{ mm}}{60 \text{ mm}}(300 \text{ rpm})$$
$$= 120 \text{ rpm}$$

$\omega_C = 120$ rpm

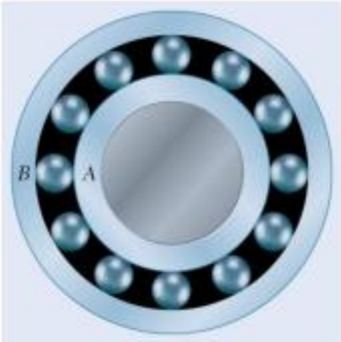
Point 2 (Point of contact of $B$ and $C$)

$$v_2 = r_B\omega_B = r_2\omega_C$$
$$\omega_B = \frac{r_2}{r_B}\omega_C$$
$$= \frac{r_2}{r_B}\left(\frac{r_A}{r_1}\right)\omega_A$$
$$= \frac{55 \text{ mm}}{24 \text{ mm}}\left(\frac{24 \text{ mm}}{60 \text{ mm}}\right)300 \text{ rpm}$$
$$\omega_B = 275 \text{ mm}$$

$\omega_B = 275$ rpm

**Ep 23.**

In the simplified sketch of a ball bearing shown, the diameter of the inner race $A$ is 60 mm and the diameter of each ball is 12 mm. The outer race $B$ is stationary while the inner race has an angular velocity of 3600 rpm. Determine ($a$) the speed of the center of each ball, ($b$) the angular velocity of each ball, ($c$) the number of times per minute each ball describes a complete circle.




## SOLUTION

Data: $\omega_A = 3600 \text{ rpm} = 376.99 \text{ rad/s}, \quad \omega_B = 0$

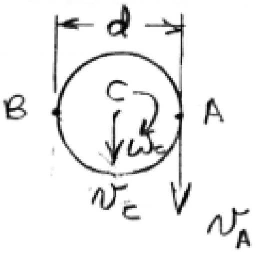
$$r_A = \frac{1}{2}d_A = 30 \text{ mm}$$

$$d = \text{diameter of ball} = 12 \text{ mm}$$

Velocity of point on inner race in contact with a ball.

$$v_A = r_A\omega_A = (30)(376.99) = 11310 \text{ mm/s}$$

Consider a ball with its center at Point $C$.




$$v_A = v_B + v_{A/B}$$

$$v_A = 0 + \omega_C d$$

$$\omega_C = \frac{v_A}{d} = \frac{11310}{12}$$

$$= 942.48 \text{ rad/s}$$

$$v_C = v_B + v_{C/B}$$

$$= 0 + \frac{1}{2}d\omega = (6)(942.48) = 5654.9 \text{ mm/s}$$

($a$) $\quad v_C = 5.65 \text{ m/s}$

($b$) Angular velocity of ball.

$$\omega_C = 942.48 \text{ rad/s} \qquad \omega_C = 9000 \text{ rpm}$$

($c$) Distance traveled by center of ball in 1 minute.

$$l_C = v_C t = 5654.9(60) = 339290 \text{ mm}$$

Circumference of circle: $\quad 2\pi r = 2\pi(30+6)$

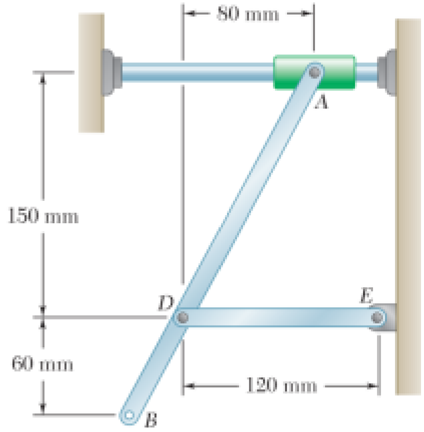
$$= 226.19 \text{ mm}$$

Number of circles completed in 1 minute:

$$n = \frac{l}{2\pi r} = \frac{339290}{226.19} \qquad n = 1500$$

**Ep 24.**

Knowing that at the instant shown the angular velocity of rod *DE* is 2.4 rad/s clockwise, determine (*a*) the velocity *A*, (*b*) the velocity of Point *B*.




## SOLUTION

Rod*DE*: Point *E* is fixed. $\boldsymbol{\omega}_{DE} = 2.4 \text{ rad} \circlearrowright$

$$v_D = \omega_{DB} r_{DE} = (2.4 \text{ rad/s})(120 \text{ mm}) = 288 \text{ mm/s}$$

$$\mathbf{v}_D = 288 \text{ mm/s} \uparrow = (288 \text{ mm/s})\mathbf{j}$$

Rod *ADB*: $\mathbf{r}_{A/D} = (80 \text{ mm})\mathbf{i} + (150 \text{ mm})\mathbf{j}, \ \boldsymbol{\omega}_{AD} = \boldsymbol{\omega}_{AD}\mathbf{k}, \quad \mathbf{v}_A = v_A\mathbf{i}$

$$\mathbf{v}_A = \mathbf{v}_D + \mathbf{v}_{D/A} = \mathbf{v}_D + \omega_{AD}\mathbf{k} \times \mathbf{r}_{A/D}$$

$$v_A\mathbf{i} = (288 \text{ mm/s})\mathbf{j} + \omega_{AD}\mathbf{k} \times [(80 \text{ mm})\mathbf{i} + (150 \text{ mm})\mathbf{j}]$$

$$v_A\mathbf{i} = 288\mathbf{j} + 80\omega_{AD}\mathbf{j} - 150\omega_{AD}\mathbf{i}$$

Equate components.

$\mathbf{i}$: $\quad v_A = -150\omega_{AD}$

$\mathbf{j}$: $\quad 0 = 288 + 80\omega_{AD}$

From Eq. (2), $\omega_{AD} = -\dfrac{288}{80} \quad \boldsymbol{\omega}_{AD} = (-3.6 \text{ rad/s})\mathbf{k}$

From Eq. (1), $v_A = -(150)(-3.6) = 540 \text{ mm/s}$

(*a*) *Velocity of collar A.* $\mathbf{v}_A = 540 \text{ mm/s} \longrightarrow$

(*b*) *Velocity of Point B.*

By proportions $\mathbf{r}_{B/D} = -\dfrac{60}{150}\mathbf{r}_{A/D} = -(32 \text{ mm})\mathbf{i} - 60 \text{ mm } \mathbf{j}$

$$\mathbf{v}_B = \mathbf{v}_D + \mathbf{v}_{B/D} = \mathbf{v}_D + \boldsymbol{\omega}_{AD} \times \mathbf{r}_{B/D}$$

$$= (288 \text{ mm/s})\mathbf{j} + [-(3.6 \text{ rad/s})\mathbf{k}] \times [-(32 \text{ mm})\mathbf{i} - (60 \text{ mm})\mathbf{j}]$$

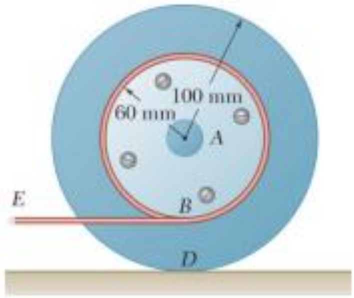
$$= (288 \text{ mm/s})\mathbf{j} + (115.2 \text{ mm/s})\mathbf{j} - (216 \text{ mm/s})\mathbf{i}$$

$$\mathbf{v}_B = -(216 \text{ mm/s})\mathbf{i} + (403.2 \text{ mm/s})\mathbf{j}$$

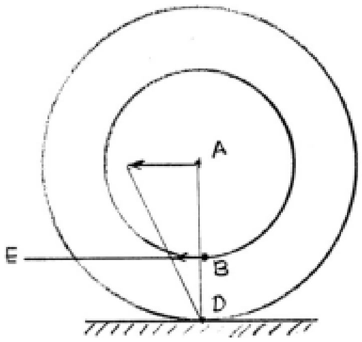
$\mathbf{v}_B = 457 \text{ mm/s} \measuredangle 61.8°$

**Ep 25.**

A 60-mm-radius drum is rigidly attached to a 100-mm-radius drum as shown. One of the drums rolls without sliding on the surface shown, and a cord is wound around the other drum. Knowing that end $E$ of the cord is pulled to the left with a velocity of 120 mm/s, determine ($a$) the angular velocity of the drums, ($b$) the velocity of the center of the drums, ($c$) the length of cord wound or unwound per second.




**SOLUTION**




Since the drum rolls without sliding, its instantaneous center lies at $D$.

$$\mathbf{v}_E = \mathbf{v}_B = 120 \text{ mm/s} \longleftarrow$$

$$v_A = v_{A/D}\omega, \quad v_B = r_{B/D}\omega$$

($a$)
$$\omega = \frac{v_B}{v_{B/D}} = \frac{120}{100-60} = 3 \text{ rad/s}$$

$$\boldsymbol{\omega} = 3.00 \text{ rad/s} \circlearrowright$$

($b$)
$$v_A = (100)(3) = 300 \text{ mm/s}$$

$$\mathbf{v}_A = 300 \text{ mm/s} \longleftarrow$$

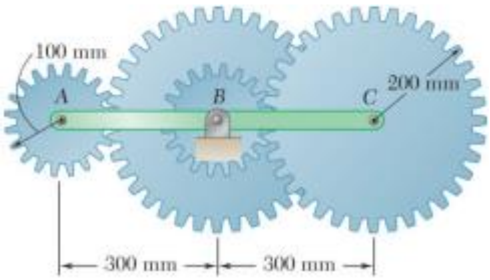
Since $v_A$ is greater than $v_B$, cord is being wound.

$$v_A - v_B = 300 - 120 = 180 \text{ mm/s}$$

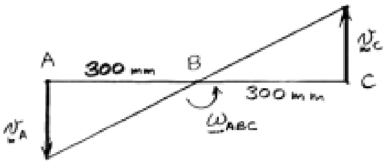
($c$) Cord wound per second = 180.0 mm

## Ep 26.

The arm $ABC$ rotates with an angular velocity of 4 rad/s counterclockwise. Knowing that the angular velocity of the intermediate gear $B$ is 8 rad/s counterclockwise, determine (*a*) the instantaneous centers of rotation of gears $A$ and $C$, (*b*) the angular velocities of gears $A$ and $C$.




### SOLUTION




*Contact points*:

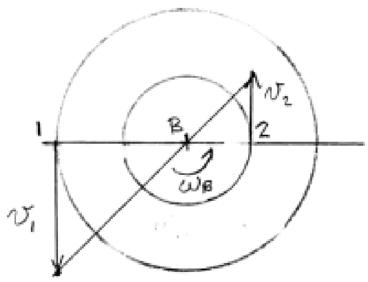
1 between gears $A$ and $B$.

2 between gears $B$ and $C$.

*Arm ABC*: $\omega_{ABC} = 4 \text{ rad/s}$ ↺

$$v_A = (0.300)(4) = 1.2 \text{ m/s} \downarrow$$

$$v_C = (0.300)(4) = 1.2 \text{ m/s} \uparrow$$

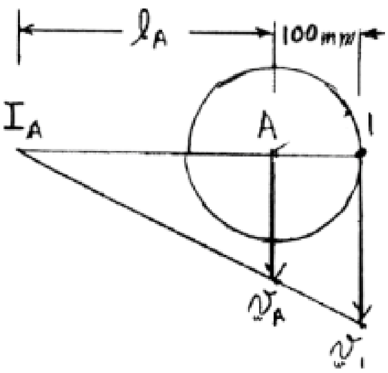



*Gear B*: $\omega_B = 8 \text{ rad/s}$ ↺

$$v_1 = (0.200)(8) = 1.6 \text{ m/s} \downarrow$$

$$v_2 = (0.100)(8) = 0.8 \text{ m/s} \uparrow$$

*Gear A*:



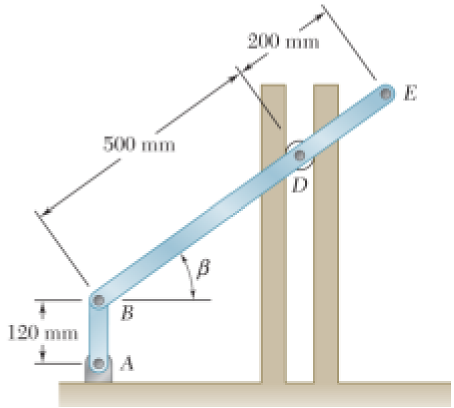

$$\omega_A = \frac{v_1 - v_A}{0.100} = \frac{1.6 - 1.2}{0.100}$$

$$\omega_A = 4 \text{ rad/s} \circlearrowright$$

$$\ell_A = \frac{v_A}{\omega_A} = \frac{1.2}{4} = 0.3 \text{ m} = 300 \text{ mm}$$

## Ep 27.

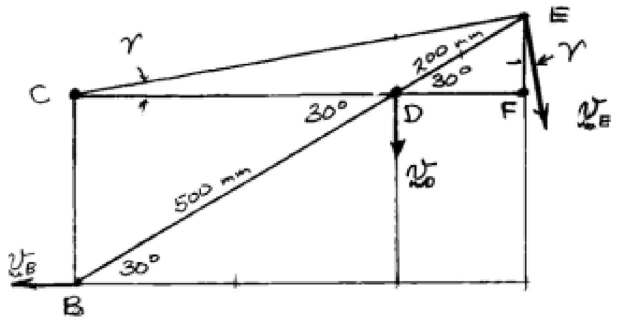
Rod *BDE* is partially guided by a roller at *D* which moves in a vertical track. Knowing that at the instant shown $\beta = 30°$, Point *E* has a velocity of 2 m/s down and to the right, determine the angular velocities of rod *BDE* and crank *AB*.




### SOLUTION

Crank*AB*: When *AB* is vertical, the velocity $\mathbf{v}_B$ at Point *B* is horizontal.

Rod *BDE*: Draw a diagram of the geometry of the rod and note that $\mathbf{v}_B$ is horizontal and $\mathbf{v}_D$ is vertical.




Locate Point *C*, the instantaneous center *C*, by noting that *CB* is vertical and *CD* is horizontal. From the diagram, with Point *F* added,

$$CF = 700\cos 30° \text{ mm} \qquad FE = 200\sin 30° \text{ mm}$$

$$CE = \sqrt{(CF)^2 + (FE)^2} = 614.41 \text{ mm} = 0.61441 \text{ m}$$

*Angular velocity of rod BDE*

$$\omega_{BDE} = \frac{v_E}{(CE)} = \frac{2 \text{ m/s}}{0.61441 \text{ m}} = 3.2552 \text{ rad/s}$$

$$\boldsymbol{\omega}_{BDE} = 3.26 \text{ rad/s} \circlearrowright$$

*Velocity of B.* $CB = 500\sin 30° \text{ mm} = 250 \text{ mm} = 0.250 \text{ m}$

$$v_B = (CB)\omega_{BDE} = (0.250)(3.2552)$$

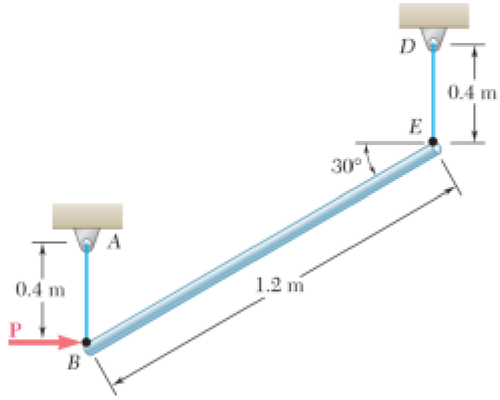
$$\mathbf{v}_B = 0.81379 \text{ m/s} \longleftarrow$$

*Angular velocity of crank AB*: $AB = 120 \text{ mm} = 0.120 \text{ m}$

$$\omega_{AB} = \frac{v_B}{(AB)} = \frac{0.81379 \text{ m/s}}{0.120 \text{ m}} \qquad \boldsymbol{\omega}_{AB} = 6.78 \text{ rad/s} \circlearrowleft$$

## Ep 28.

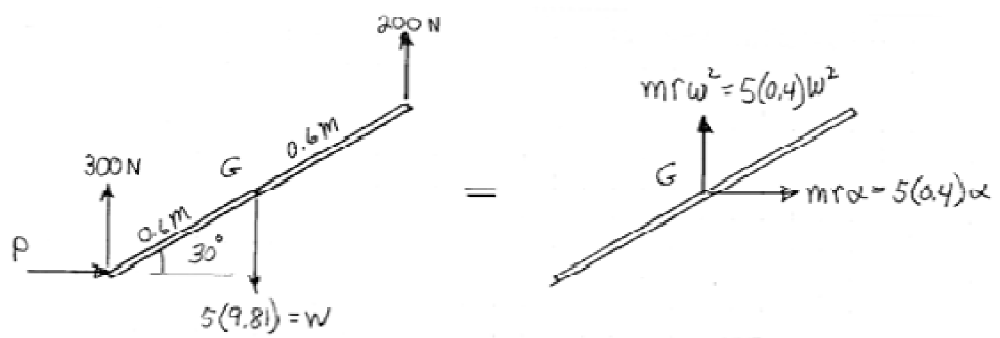
At the instant shown the tensions in the vertical ropes *AB* and *DE* are 300 N and 200 N, respectively. Knowing that the mass of the uniform bar *BE* is 5 kg, determine, at this instant, (*a*) the force **P**, (*b*) the magnitude of the angular velocity of each rope, (*c*) the angular acceleration of each rope.




**SOLUTION**

Given: $T_{AB} = 300\text{ N},\ T_{DE} = 200\text{ N},\ m_{BE} = 5\text{ kg}$

Free Body Diagram:




Kinetics:

$$\circlearrowleft + \sum M_G = 0 = 200(0.6)(0.866) - 300(0.6)(0.866) + P(0.6)(0.5)$$

(*a*) **P** $= 173.2\text{ N} \longrightarrow$

$$+\uparrow \sum F_y = 300 - 5(9.81) + 200 = 5(0.4)\omega^2$$

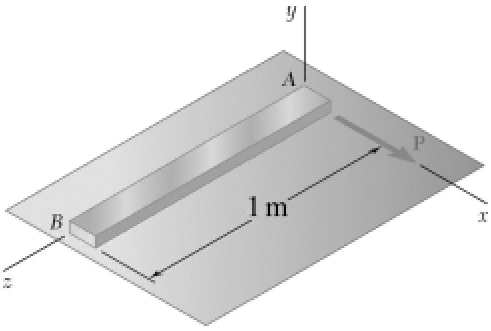
(*b*) $|\omega| = 15.02\text{ rad/s}$

$$\xrightarrow{+} \sum F_x = P = 5(0.4)\alpha = 173.2$$

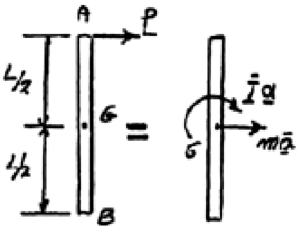
(*c*) $\boldsymbol{\alpha} = 86.6\text{ rad/s}^2 \circlearrowleft$

**Ep 29.**

A uniform slender rod $AB$ rests on a frictionless horizontal surface, and a force **P** of magnitude 1 N is applied at $A$ in a direction perpendicular to the rod. Knowing that the rod weighs 9 N, determine the acceleration of ($a$) Point $A$, ($b$) Point $B$.




**SOLUTION**




$$m = \frac{W}{g}$$

$$I = \frac{1}{12}\frac{W}{g}L^2$$

$$\xrightarrow{+} \Sigma F_x = \Sigma (F_x)_{\text{eff}}: \quad P = m\bar{a} = \frac{W}{g}\bar{a}$$

$$\bar{a} = \frac{P}{W}g = \frac{1\text{ N}}{9\text{ N}}g = \frac{g}{9} \qquad \bar{\mathbf{a}} = \frac{1}{9}g \rightarrow$$

$$\circlearrowright \Sigma M_G = \Sigma (M_G)_{\text{eff}}: \quad P\frac{L}{2} = \bar{I}\alpha = \frac{1}{12}\frac{W}{g}L^2\alpha$$

$$\alpha = 6\frac{P}{W}\frac{g}{L} = 6\frac{1\text{ N}}{9\text{ N}}\frac{g}{L} \qquad \boldsymbol{\alpha} = \frac{2}{3}\frac{g}{L} \circlearrowright$$

($a$) Acceleration of Point $A$.

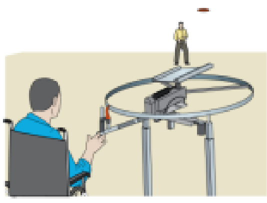
$$\xrightarrow{+} \mathbf{a}_A = \bar{\mathbf{a}} + \frac{L}{2}\alpha = \frac{g}{9} + \frac{L}{2}\cdot\frac{2}{3}\frac{g}{L} = g\left(\frac{4}{9}\right) = 4.36\text{ m/s}^2) \qquad \mathbf{a}_A = 4.36\text{ m/s}^2 \rightarrow$$

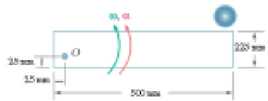
($b$) Acceleration of Point $B$.

$$\xrightarrow{+} \mathbf{a}_B = \bar{\mathbf{a}} - \frac{L}{2}\alpha = \frac{g}{9} - \frac{L}{2}\cdot\frac{2}{3}\frac{g}{L} = -\frac{2g}{9} = -\frac{2}{9}(9.81\text{ m/s}^2) \qquad \mathbf{a}_B = 2.18\text{ m/s}^2 \leftarrow$$

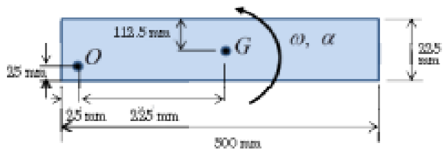
## Ep 30.

An adapted launcher uses a torsional spring about point $O$ to help people with mobility impairments throw a Frisbee. Just after the Frisbee leaves the arm, the angular velocity of the throwing arm is 200 rad/s and its acceleration is 10 rad/s$^2$, both counterclockwise. The rotation point $O$ is located 25 mm. from the two sides. Assume that you can model the 1-kg throwing arm as a uniform rectangle. Just after the Frisbee leaves the arm, determine *(a)* the moment about $O$ caused by the spring, *(b)* the forces on the pin at $O$.





### SOLUTION




Given: $m = 1\text{kg}$

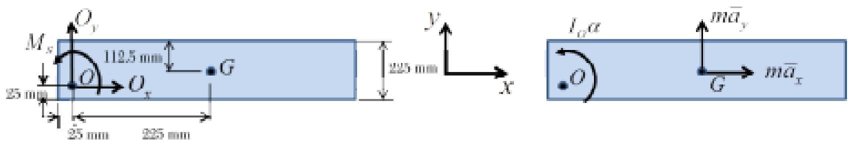
$\omega = 200 \text{ rad/s}, \ \alpha = 10 \text{ rad/s}^2$

Mass Moment of Inertia: $I_O = I_G + md_{G/O}^2$

$$= \frac{1}{12}(1\text{ kg})[(0.225)^2 + (0.5)^2] + (1\text{ kg})[(0.225)^2 + (0.0875)^2]$$

$$= \frac{1}{2}\text{ kg}\cdot\text{m}^2$$

Free Body Diagram:




Kinetics:

$$\sum F_x = m\bar{a}_x \qquad \sum F_y = m\bar{a}_y \qquad \sum M_O = I_O\alpha$$

$$O_x = (1)\bar{a}_x \qquad O_y = (1)\bar{a}_y \qquad M_s = \left(\frac{1}{12}\right)(10)$$

$$= \left(\frac{5}{6}\right)\text{ N}\cdot\text{m}$$

*(a)* $M_S = 0.833 \text{ N}\cdot\text{m}$

Kinematics:

$$\mathbf{a}_G = \cancel{\mathbf{a}_O} + \boldsymbol{\alpha} \times \mathbf{r}_{G/O} - \omega^2 \mathbf{r}_{G/O}$$

$$= 10\mathbf{k} \times (0.225\mathbf{i} + 0.0875\mathbf{j}) - (200)^2(0.225\mathbf{i} + 0.0875\mathbf{j})$$

$$= -8999\mathbf{i} - 3498\mathbf{j} \text{ m/s}^2$$

$$\bar{a}_x = -8999 \text{ m/s}^2 \text{ and } \bar{a}_y = -3498 \text{ m/s}^2$$

Therefore:

$$O_x = (1)(-8999) \quad O_y = 1(-3498)$$

$$= -8999 \text{ N} \qquad = -3498 \text{ N}$$

*(b)* Magnitude of Force at O: $O = \sqrt{O_x^2 + O_y^2} = \sqrt{(-8999)^2 + (-3498)^2}$

$O = 9650 \text{ N}$

**Ep 31.**

A drum of 80-mm radius is attached to a disk of 160-mm radius. The disk and drum have a combined mass of 5 kg and combined radius of gyration of 120 mm. A cord is attached as shown and pulled with a force **P** of magnitude 20 N. Knowing that the coefficients of static and kinetic friction are $\mu_s = 0.25$ and $\mu_k = 0.20$, respectively, determine (*a*) whether or not the disk slides, (*b*) the angular acceleration of the disk and the acceleration of *G*.

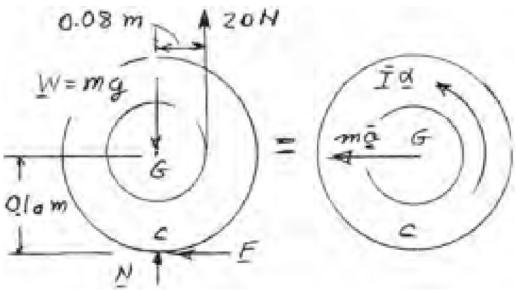

**SOLUTION**

*Assume disk rolls*:

$$\bar{a} = r\alpha = (0.16\text{ m})\alpha$$

$$\bar{I} = m\bar{k}^2 = (5\text{ kg})(0.12\text{ m})^2$$

$$= 0.072\text{ kg}\cdot\text{m}^2$$




$$+\circlearrowleft\ \Sigma M_C = \Sigma(M_C)_{\text{eff}}:\quad (20\text{ N})(0.08\text{ m}) = (m\bar{a})r + \bar{I}\alpha$$

$$1.6\text{ N}\cdot\text{m} = (5\text{ kg})(0.16\text{ m})^2\alpha + \left(0.072\text{ kg}\cdot\text{m}^2\right)\alpha$$

$$\alpha = 8\text{ rad/s}^2 \circlearrowright \qquad\qquad \text{or}\ \ \alpha = 8\text{ rad/s}^2 \circlearrowright$$

$$\bar{a} = r\alpha = (0.16\text{ m})\left(8\text{ rad/s}^2\right) = 1.28\text{ m/s}^2$$

$$\text{or}\ \ \bar{\mathbf{a}} = 1.28\text{ m/s}^2 \longleftarrow$$

$$\xleftarrow{+}\ \Sigma F_x = \Sigma(F_x)_{\text{eff}}:\quad F = m\bar{a} = (5\text{ kg})\left(1.28\text{ m/s}^2\right) = 6.40\text{ N}$$

$$+\uparrow \Sigma F_y = 0 \quad N + 20\text{ N} - mg = 0,\quad N + 20\text{ N} - (5\text{ kg})\left(9.81\text{ m/s}^2\right) = 0$$

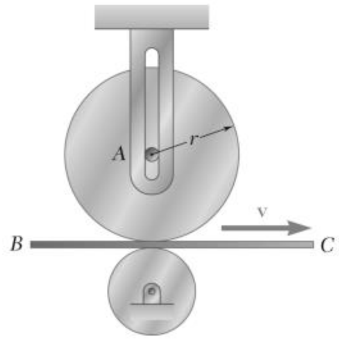
$$N = 29.05\text{ N}$$

$$F_m = \mu_s N = 0.25(29.05\text{ N}) = 7.2625\text{ N}$$

Since $F < Fm$, *disk rolls without sliding*

**Ep 32.**

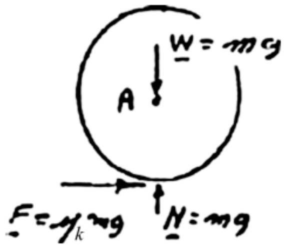
Disk $A$, of weight 5 kg and radius $r = 150$ mm, is at rest when it is placed in contact with belt $BC$, which moves to the right with a constant speed $v = 12$ m/s. Knowing that $\mu_k = 0.20$ between the disk and the belt, determine the number of revolutions executed by the disk before it attains a constant angular velocity.




**SOLUTION**

Work of external friction force on disk $A$.

Only force doing work is $F$. Since its moment about $A$ is $M = rF$, we have




$$\begin{aligned} U_{1\to2} &= M\theta \\ &= rF\theta \\ &= r(\mu_k mg)\theta \end{aligned}$$

Kinetic energy of disk $A$.

Angular velocity becomes constant when $\omega_2 = \dfrac{v}{r}$

$$T_1 = 0$$

$$\begin{aligned} T_2 &= \frac{1}{2}\bar{I}\omega_2^2 \\ &= \frac{1}{2}\left(\frac{1}{2}mr^2\right)\left(\frac{v}{r}\right)^2 \\ &= \frac{mv^2}{4} \end{aligned}$$

Principle of work and energy for disk $A$.

$$T_1 + U_{1-2} = T_2: \quad 0 + r\mu_k mg\theta = \frac{mv^2}{4}$$

Angle change $\theta = \dfrac{v^2}{4r\mu_k g}$ rad $\qquad \theta = \dfrac{v^2}{8\pi r\mu_k g}$ rev

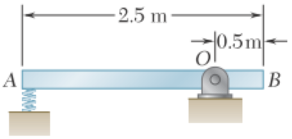
Data:

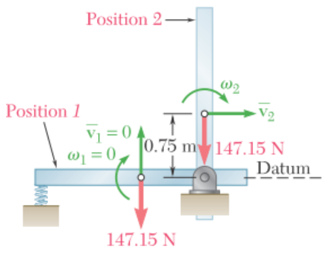
$$r = 0.15 \text{ m}$$
$$\mu_k = 0.20$$
$$v = 12 \text{ m/s}$$

$$\theta = \frac{(12 \text{ m/s})^2}{8\pi(0.15 \text{ m})(0.20)(9.81 \text{ m/s}^2)} \qquad \theta = 19.47 \text{ rev}$$

**Ep 33.**

A 15-kg slender rod $AB$ is 2.5 m long and is pivoted about a point $O$ which is 0.5 m from end $B$. The other end is pressed against a spring of constant $k$ = 300 kN/m until the spring is compressed 40 mm. The rod is then in a horizontal position. If the rod is released from this position, determine its angular velocity and the reaction at the pivot $O$ as the rod passes through a vertical position.




## SOLUTION

**Position 1.** ***Potential Energy.*** Since the spring is compressed 40 mm, we have $x_1$ = 40 mm.

$$V_e = \tfrac{1}{2}kx_1^2 = \tfrac{1}{2}(300{,}000\ \text{N/m})(0.040\text{m}^2) = 240\ \text{J}$$

Choosing the datum as shown, we have $V_g = 0$; therefore,

$$V_1 = V_e + V_g = 240\ \text{J}$$

***Kinetic Energy.*** Since the velocity in position *1* is zero, we have $T_1 = 0$.

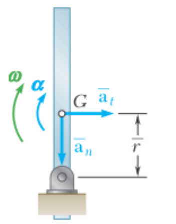
**Position 2.** ***Potential Energy.*** The elongation of the spring is zero, and we have $V_e = 0$. Since the center of gravity of the rod is now 0.75 m above the datum,

$$V_g = (147.15\ \text{N})(0.75\ \text{m}) = 110.4\ \text{J}$$
$$V_2 = V_e + V_g = 110.4\ \text{J}$$

***Kinetic Energy.*** Denoting by $\boldsymbol{\omega}_2$ the angular velocity of the rod in position 2, we note that the rod rotates about $O$ and write $\bar{v}_2 = \bar{r}\omega_2 = 0.75\omega_2$.

$$\bar{I} = \tfrac{1}{12}ml^2 = \frac{1}{12}(15\ \text{kg})(2.5\text{m})^2 = 7.81\,\text{kg-m}^2$$

$$T_2 = \tfrac{1}{2}m\bar{v}_2^2 + \tfrac{1}{2}\bar{I}\omega_2^2 = \frac{1}{2}\times(15)(0.75\,\omega_2)^2 + \frac{1}{2}\times(7.81)\omega_2^2 = 8.12\ \omega_2^2$$



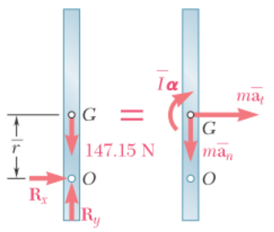

**Conservation of Energy**

$$T_1 + V_1 = T_2 + V_2$$
$$0 + 240\ \text{J} = 8.12\ \omega_2^2 + 110.4\ \text{J}$$

$$\omega_2 = 3.995\ \text{rad/s}\circlearrowright$$

**Reaction in Position 2.** Since $\omega_2$ = 3.995 rad/s, the components of the acceleration of $G$ as the rod passes through position 2 are

$$\bar{a}_n = \bar{r}\omega_2^2 = (0.75\,\text{m})(3.995\,\text{rad/s})^2 = 11.97\,\text{m/s}^2 \qquad \bar{\mathbf{a}}_n = 11.97\ \text{m/s}^2\downarrow$$
$$\bar{a}_t = \bar{r}\alpha \qquad \bar{\mathbf{a}}_t = \bar{r}\alpha \rightarrow$$




We express that the system of external forces is equivalent to the system of effective forces represented by the vector of components $m\bar{\mathbf{a}}_t$ and $m\bar{\mathbf{a}}_n$ attached at $G$ and the couple $\bar{I}\boldsymbol{\alpha}$.

$+\circlearrowright\Sigma M_O = \Sigma(M_O)_{\text{eff}}$: $\quad 0 = \bar{I}\alpha + m(\bar{r}\alpha)\bar{r} \qquad \alpha = 0$

$\xrightarrow{+}\Sigma F_x = \Sigma(F_x)_{\text{eff}}$: $\quad R_x = m(\bar{r}\alpha) \qquad R_x = 0$

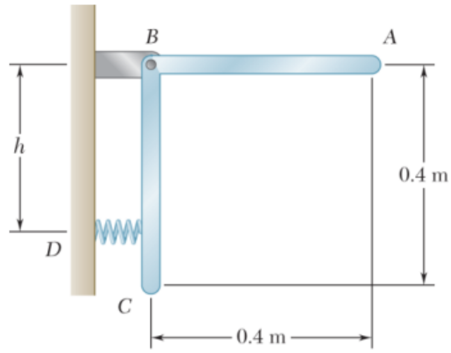
$+\uparrow\Sigma F_y = \Sigma(F_y)_{\text{eff}}$: $\quad R_y - 147.15\ \text{N} = -m\bar{a}_n$

$$R_y - 147.15\ \text{N} = -(15\ \text{kg})\,(11.97\ \text{m/s}^2)$$

$$R_y = -32.4\ \text{N} \qquad \mathbf{R} = 32.4\ \text{N}\downarrow$$

**Ep 34.**

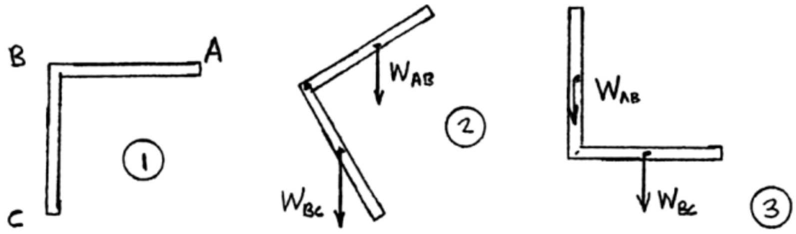
Two identical slender rods *AB* and *BC* are welded together to form an L-shaped assembly. The assembly is pressed against a spring at *D* and released from the position shown. Knowing that the maximum angle of rotation of the assembly in its subsequent motion is 90° counterclockwise, determine the magnitude of the angular velocity of the assembly as it passes through the position where rod *AB* forms an angle of 30° with the horizontal.




**SOLUTION**

Moment of inertia about *B*.

$$I_B = \frac{1}{3}m_{AB}l^2 + \frac{1}{3}m_{BC}l^2$$




*Position 2.*

$$\theta = 30°$$

$$V_2 = W_{AB}(h_{AB})_2 + W_{BC}(h_{BC})_2$$

$$= W_{AB}\frac{l}{2}\sin 30° + W_{BC}\left(-\frac{l}{2}\cos 30°\right)$$

$$T_2 = \frac{1}{2}I_B\omega_2^2 = \frac{1}{6}(m_{AB}+m_{BC})l^2\omega_2^2$$

*Position 3.*

$$\theta = 90°$$

$$V_3 = W_{AB}\frac{l}{2} \quad T_3 = 0$$

Conservation of energy.

$$T_2 + V_2 = T_3 + V_3:$$

$$\frac{1}{6}(m_{AB}+m_{BC})l^2\omega_2^2 + W_{AB}\frac{l}{2}\sin 30° - W_{BC}\frac{l}{2}\cos 30° = 0 + W_{AB}\frac{l}{2}$$

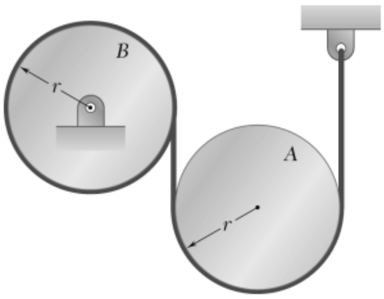
$$\omega_2^2 = \frac{3}{l}\cdot\frac{W_{AB}(1-\sin 30°) + W_{BC}\cos 30°}{m_{AB}+m_{BC}}$$

$$= \frac{3}{2}\frac{g}{l}[1 - \sin 30° + \cos 30°]$$

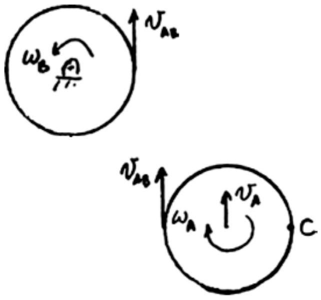
$$= 2.049\frac{g}{l} = 2.049\frac{9.81}{0.4} = 50.25 \qquad \omega_2 = 7.09 \text{ rad/s}$$

## Ep 35.

Two uniform cylinders, each of mass $m = 7$ kg and radius $r = 100$ mm are connected by a belt as shown. Knowing that the initial angular velocity of cylinder $B$ is 30 rad/s counterclockwise, determine (*a*) the distance through which cylinder $A$ will rise before the angular velocity of cylinder $B$ is reduced to 5 rad/s, (*b*) the tension in the portion of belt connecting the two cylinders.




**SOLUTION**




Kinematics.
$$v_{AB} = r\omega_B$$

Point $C$ is the instantaneous center of cylinder $A$.

$$\omega_A = \frac{v_{AB}}{2r} = \frac{1}{2}\omega_B$$

$$\bar{v}_A = r\omega_A = \frac{1}{2}r\omega_B$$

Moment of inertia.
$$\bar{I} = \frac{1}{2}\frac{W}{g}r^2$$

Kinetic energy.

Cyl $B$:
$$\frac{1}{2}\bar{I}\omega_B^2 = \frac{1}{2}\left(\frac{1}{2}\frac{W}{g}r^2\right)\omega_B^2 = \frac{1}{4}\frac{W}{g}r^2\omega_B^2$$

Cyl $A$:
$$\frac{1}{2}mv_A^2 + \frac{1}{2}\bar{I}\omega_A^2 = \frac{1}{2}\frac{W}{g}\left(\frac{1}{2}r\omega_B\right)^2 + \frac{1}{2}\left(\frac{1}{2}\frac{W}{g}r^2\right)\left(\frac{1}{2}\omega_B\right)^2$$
$$= \frac{3}{16}\frac{W}{g}r^2\omega_B^2$$

Total:
$$T = \frac{7}{16}\frac{W}{g}r^2\omega_B^2$$

(*a*) *Distance h that cylinder A will rise.*

Conservation of energy for system.

$$T_1 + V_1 = T_2 + V_2: \quad \frac{7}{16}\frac{W}{g}r^2(\omega_B)_1^2 + 0 = \frac{7}{16}\frac{W}{g}r^2(\omega_B)_2^2 + Wh$$

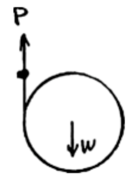
$$h = \frac{7}{16}\frac{r^2}{g}[(\omega_B)_1^2 - (\omega_B)_2]^2$$

$$= \left(\frac{7}{16}\right)\frac{(0.1)^2}{(9.81)}(30^2 - 5^2)$$

$$= 0.3902 \text{ m} \qquad h = 0.390 \text{ m}$$

(*b*) *Tension in belt between the cylinders.*

When cylinder *A* moves up a distance *h*, the belt moves up a distance 2*h*.




Work: $U_{1\to 2} = P(2h) - Wh$

Principle of work and energy for cylinder *A*.

$$T_1 + U_{1\to 2} = T_2: \quad \frac{3}{16}\frac{W}{g}r^2(\omega_B)_1^2 + 2Ph - Wh = \frac{3}{16}\frac{W}{g}r^2(\omega_B)_2^2$$

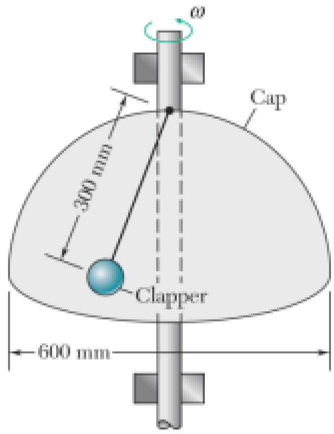
$$P = \frac{1}{2}W - \frac{3}{32}\frac{Wr^2}{gh}\left[(\omega_B)_1^2 - (\omega_B)_2^2\right]$$

$$= \frac{1}{2}W - \frac{3}{14}W$$

$$= \frac{2}{7}W = \frac{2}{7}(7)(9.81) \qquad P = 19.62 \text{ N}$$

**Ep 36.**

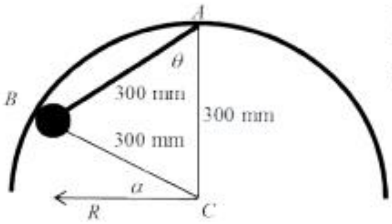
A spherical-cap governor is fixed to a vertical shaft that rotates with angular velocity $\omega$. When the string-supported clapper of mass $m$ touches the cap, a cutoff switch is operated electrically to reduce the speed of the shaft. Knowing that the radius of the clapper is small relative to the cap, determine the minimum angular speed at which the cutoff switch operates.




**SOLUTION**

Given: $\omega_{min}$ is the angular velocity when the clapper hits the cap

Geometry From Figure: Distance AB is 300 mm, the length of the Clapper Arm

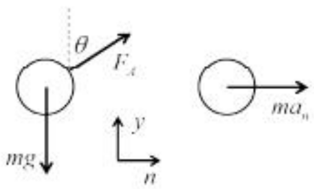



Distance BC and AC is 300 mm, the radius of the Spherical Cap.

Therefore $\triangle ABC$ is equilateral, so $\theta = 60°$ and $\alpha = 30°$

$R = 0.3 * \cos\alpha$

Free Body Diagram of Clapper:




Equations of Motion:

$$\sum F_y = ma_y$$

$$\mathrm{F}_A \cos\theta - mg = m(0)$$

$$F_A = \frac{mg}{\cos\theta} \qquad (1)$$

$$\sum F_n = ma_n$$

$$F_A \sin\theta = mR\omega_{min}^2$$

$$\omega_{win} = \sqrt{\frac{F_A \sin\theta}{mR}} \qquad (2)$$

Substitute (1)into (2):

$$\omega_{win} = \sqrt{\frac{\cancel{m} g \tan\theta}{\cancel{m} R}}$$

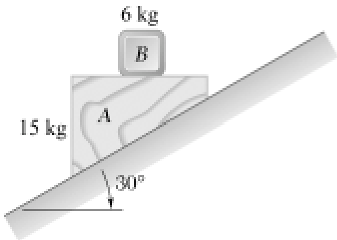
$$= \sqrt{\frac{9.81 \tan 60°}{0.3 \cos 30°}}$$

$$= 8.087 \text{ rad/s}$$

$\omega_{win}$ =77.23 rpm

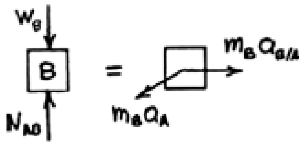
**Ep37.**

A 6-kg block $B$ rests as shown on the upper surface of a 15 kg wedge $A$. Neglecting friction, determine immediately after the system is released from rest ($a$) the acceleration of $A$, ($b$) the acceleration of $B$ relative to $A$.




**SOLUTION**

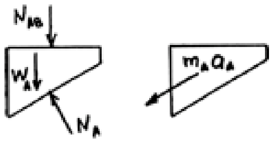
Acceleration vectors:




$$\mathbf{a}_A = a_A \swarrow 30^\circ, \quad \mathbf{a}_{B/A} = a_{B/A} \longrightarrow$$

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

Block $B$: $\quad \xrightarrow{+} \Sigma F_x = ma_x: \quad m_B a_{B/A} - m_B a_A \cos 30^\circ = 0$




$$a_{B/A} = a_A \cos 30^\circ \tag{1}$$

$$+\downarrow \Sigma F_y = ma_y: \quad N_{AB} - W_B = -m_B a_A \sin 30^\circ$$

$$N_{AB} = W_B - (W_B \sin 30^\circ)\frac{a_A}{g} \tag{2}$$

Block $A$: $\quad +\swarrow \Sigma F = ma: \quad W_A \sin 30^\circ + N_{AB} \sin 30^\circ = W_A \dfrac{a_A}{g}$

$$W_A \sin 30^\circ + W_B \sin 30^\circ - (W_B \sin^2 30^\circ)\frac{a_A}{g} = W_A \frac{a_A}{g}$$

$$a_A = \frac{(W_A + W_B)\sin 30^\circ}{W_A + W_B \sin^2 30^\circ} g = \frac{(m_A + m_B)\sin 30^\circ}{m_A + m_B \sin^2 30^\circ} g$$

$$= \frac{21 \sin 30^\circ}{15 + 6\sin^2 30^\circ} \times (9.81)$$

($a$) $\quad \mathbf{a}_A = 6.24 \text{ m/s}^2 \swarrow 30^\circ$

$$a_{B/A} = (6.24)\cos 30^\circ = 5.40 \text{ m/s}^2$$

($b$) $\quad \mathbf{a}_{B/A} = 5.40 \text{ m/s}^2 \longrightarrow$